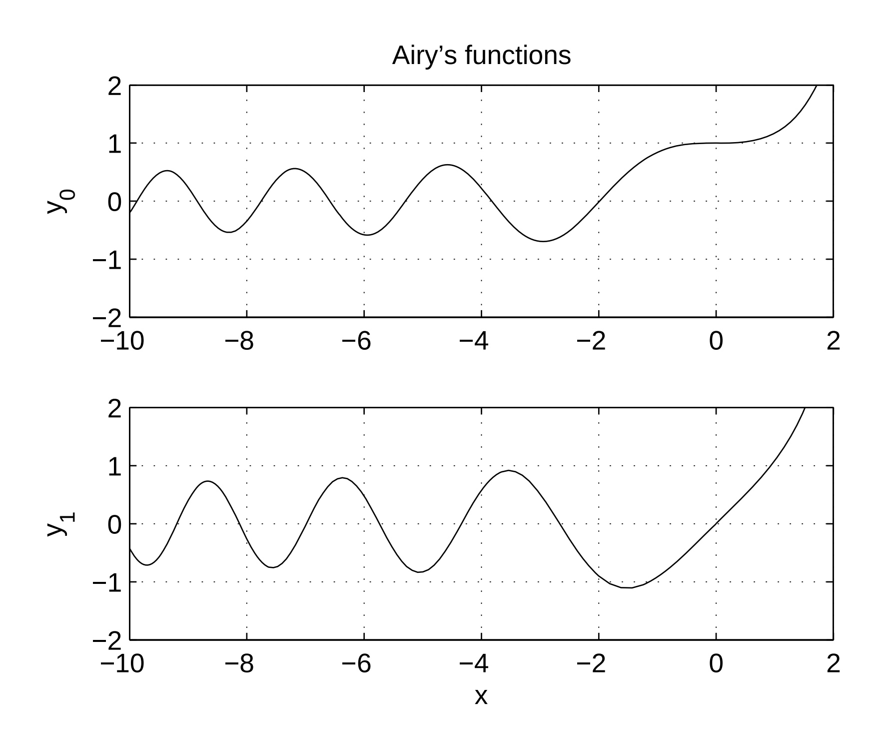
Introduction to Differential Equations

Lecture notes for MATH 2351/2352

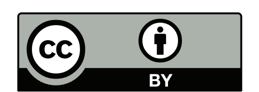
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Preface

What follows are my lecture notes for a first course in differential equations, taught at the Hong Kong University of Science and Technology. Included in these notes are links to short tutorial videos posted on YouTube.

Much of the material of Chapters 2-6 and 8 has been adapted from the widely used textbook "Elementary differential equations and boundary value problems" by Boyce & DiPrima (John Wiley & Sons, Inc., Seventh Edition, ©2001). Many of the examples presented in these notes may be found in this book. The material of Chapter 7 is adapted from the textbook "Nonlinear dynamics and chaos" by Steven H. Strogatz (Perseus Publishing, ©1994).

All web surfers are welcome to download these notes, watch the YouTube videos, and to use the notes and videos freely for teaching and learning. An associated free review book with links to YouTube videos is also available from the ebook publisher bookboon.com. I welcome any comments, suggestions or corrections sent by email to jeffrey.chasnov@ust.hk. Links to my website, these lecture notes, my YouTube page, and the free ebook from bookboon.com are given below.

Homepage:

http://www.math.ust.hk/~machas

YouTube:

https://www.youtube.com/user/jchasnov

Lecture notes:

http://www.math.ust.hk/~machas/differential-equations.pdf

Bookboon:

http://bookboon.com/en/differential-equations-with-youtube-examples-ebook

Contents

Chapter 0

A short mathematical review

A basic understanding of calculus is required to undertake a study of differential equations. This zero chapter presents a short review.

0.1 The trigonometric functions

The Pythagorean trigonometric identity is

$$\sin^2 x + \cos^2 x = 1,$$

and the addition theorems are

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

Also, the values of $\sin x$ in the first quadrant can be remembered by the rule of quarters, with $0^\circ = 0$, $30^\circ = \pi/6$, $45^\circ = \pi/4$, $60^\circ = \pi/3$, $90^\circ = \pi/2$:

$$\sin 0^\circ = \sqrt{\frac{0}{4}}, \qquad \sin 30^\circ = \sqrt{\frac{1}{4}}, \qquad \sin 45^\circ = \sqrt{\frac{2}{4}},$$
$$\sin 60^\circ = \sqrt{\frac{3}{4}}, \qquad \sin 90^\circ = \sqrt{\frac{4}{4}}.$$

The following symmetry properties are also useful:

$$\sin(\pi/2 - x) = \cos x, \quad \cos(\pi/2 - x) = \sin x;$$

and

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x).$$

0.2 The exponential function and the natural logarithm

The transcendental number e, approximately 2.71828, is defined as

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

The exponential function $\exp(x) = e^x$ and natural logarithm $\ln x$ are inverse functions satisfying

$$e^{\ln x} = x, \quad \ln e^x = x.$$

The usual rules of exponents apply:

$$e^x e^y = e^{x+y}, \quad e^x/e^y = e^{x-y}, \quad (e^x)^p = e^{px}.$$

The corresponding rules for the logarithmic function are

$$\ln(xy) = \ln x + \ln y, \quad \ln(x/y) = \ln x - \ln y, \quad \ln x^p = p\ln x.$$

0.3 Definition of the derivative

The derivative of the function $y = f(x)$, denoted as $f'(x)$ or dy/dx, is defined as the slope of the tangent line to the curve $y = f(x)$ at the point (x, y). This slope is obtained by a limit, and is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{1}$$

0.4 Differentiating a combination of functions

0.4.1 The sum or difference rule

The derivative of the sum of $f(x)$ and $g(x)$ is

$$(f+g)' = f' + g'.$$

Similarly, the derivative of the difference is

$$(f-g)' = f' - g'.$$

0.4.2 The product rule

The derivative of the product of $f(x)$ and $g(x)$ is

$$(fg)' = f'g + fg',$$

and should be memorized as "the derivative of the first times the second plus the first times the derivative of the second."

0.4.3 The quotient rule

The derivative of the quotient of $f(x)$ and $g(x)$ is

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2},$$

and should be memorized as "the derivative of the top times the bottom minus the top times the derivative of the bottom over the bottom squared."

0.4.4 The chain rule

The derivative of the composition of $f(x)$ and $g(x)$ is

$$\Big(f(g(x))\Big)' = f'(g(x)) \cdot g'(x),$$

and should be memorized as "the derivative of the outside times the derivative of the inside."

0.5 Differentiating elementary functions

0.5.1 The power rule

The derivative of a power of x is given by

$$\frac{d}{dx}x^p = px^{p-1}.$$

0.5.2 Trigonometric functions

The derivatives of $\sin x$ and $\cos x$ are

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x.$$

We thus say that "the derivative of sine is cosine," and "the derivative of cosine is minus sine." Notice that the second derivatives satisfy

$$(\sin x)'' = -\sin x, \quad (\cos x)'' = -\cos x.$$

0.5.3 Exponential and natural logarithm functions

The derivative of e^x and $\ln x$ are

$$(e^x)' = e^x, \quad (\ln x)' = \frac{1}{x}.$$

0.6 Definition of the integral

The definite integral of a function $f(x) > 0$ from $x = a$ to b ($b > a$) is defined as the area bounded by the vertical lines $x = a$, $x = b$, the x-axis and the curve $y = f(x)$. This "area under the curve" is obtained by a limit. First, the area is approximated by a sum of rectangle areas. Second, the integral is defined to be the limit of the rectangle areas as the width of each individual rectangle goes to zero and the number of rectangles goes to infinity. This resulting infinite sum is called a *Riemann Sum*, and we define

$$\int_a^b f(x)dx = \lim_{h \to 0} \sum_{n=1}^{N} f(a + (n-1)h) \cdot h, \tag{2}$$

where $N = (b - a)/h$ is the number of terms in the sum. The symbols on the left-hand-side of (2) are read as "the integral from a to b of f of x dee x." The Riemann Sum definition is extended to all values of a and b and for all values of $f(x)$ (positive and negative). Accordingly,

$$\int_b^a f(x)dx = -\int_a^b f(x)dx \quad \text{and} \quad \int_a^b (-f(x))dx = -\int_a^b f(x)dx.$$

Also,

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx,$$

which states when $f(x) > 0$ and $a < b < c$ that the total area is equal to the sum of its parts.

0.7 The fundamental theorem of calculus

View tutorial on YouTube

Using the definition of the derivative, we differentiate the following integral:

$$\begin{aligned}\frac{d}{dx}\int_a^x f(s)ds &= \lim_{h\to 0}\frac{\int_a^{x+h} f(s)ds - \int_a^x f(s)ds}{h}\\ &= \lim_{h\to 0}\frac{\int_x^{x+h} f(s)ds}{h}\\ &= \lim_{h\to 0}\frac{hf(x)}{h}\\ &= f(x).\end{aligned}$$

This result is called the fundamental theorem of calculus, and provides a connection between differentiation and integration.

The fundamental theorem teaches us how to integrate functions. Let $F(x)$ be a function such that $F'(x) = f(x)$. We say that $F(x)$ is an antiderivative of $f(x)$. Then from the fundamental theorem and the fact that the derivative of a constant equals zero,

$$F(x) = \int_a^x f(s)ds + c.$$

Now, $F(a) = c$ and $F(b) = \int_a^b f(s)ds + F(a)$. Therefore, the fundamental theorem shows us how to integrate a function $f(x)$ provided we can find its antiderivative:

$$\int_a^b f(s)ds = F(b) - F(a). \tag{3}$$

Unfortunately, finding antiderivatives is much harder than finding derivatives, and indeed, most complicated functions cannot be integrated analytically.

We can also derive the very important result (3) directly from the definition of the derivative (1) and the definite integral (2). We will see it is convenient to choose the same h in both limits. With $F'(x) = f(x)$, we have

$$\begin{aligned}\int_a^b f(s)ds &= \int_a^b F'(s)ds\\ &= \lim_{h\to 0}\sum_{n=1}^{N} F'(a+(n-1)h)\cdot h\\ &= \lim_{h\to 0}\sum_{n=1}^{N}\frac{F(a+nh)-F(a+(n-1)h)}{h}\cdot h\\ &= \lim_{h\to 0}\sum_{n=1}^{N} F(a+nh)-F(a+(n-1)h).\end{aligned}$$

The last expression has an interesting structure. All the values of $F(x)$ evaluated at the points lying between the endpoints a and b cancel each other in consecutive terms. Only the value $-F(a)$ survives when $n = 1$, and the value $+F(b)$ when $n = N$, yielding again (3).

0.8 Definite and indefinite integrals

The Riemann sum definition of an integral is called a *definite integral*. It is convenient to also define an indefinite integral by

$$\int f(x)dx = F(x),$$

where F(x) is the antiderivative of $f(x)$.

0.9 Indefinite integrals of elementary functions

From our known derivatives of elementary functions, we can determine some simple indefinite integrals. The power rule gives us

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1.$$

When $n = -1$, and x is positive, we have

$$\int \frac{1}{x} dx = \ln x + c.$$

If x is negative, using the chain rule we have

$$\frac{d}{dx} \ln(-x) = \frac{1}{x}.$$

Therefore, since

$$|x| = \begin{cases} -x & \text{if } x < 0; \\ x & \text{if } x > 0, \end{cases}$$

we can generalize our indefinite integral to strictly positive or strictly negative x:

$$\int \frac{1}{x} dx = \ln|x| + c.$$

Trigonometric functions can also be integrated:

$$\int \cos x dx = \sin x + c, \quad \int \sin x dx = -\cos x + c.$$

Easily proved identities are an addition rule:

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx;$$

and multiplication by a constant:

$$\int Af(x)dx = A \int f(x)dx.$$

This permits integration of functions such as

$$\int (x^2 + 7x + 2)dx = \frac{x^3}{3} + \frac{7x^2}{2} + 2x + c,$$

and

$$\int (5\cos x + \sin x)dx = 5\sin x - \cos x + c.$$

0.10 Substitution

More complicated functions can be integrated using the chain rule. Since

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x),$$

we have

$$\int f'(g(x)) \cdot g'(x) dx = f(g(x)) + c.$$

This integration formula is usually implemented by letting $y = g(x)$. Then one writes $dy = g'(x)dx$ to obtain

$$\begin{aligned}\int f'(g(x))g'(x)dx &= \int f'(y)dy \\ &= f(y) + c \\ &= f(g(x)) + c.\end{aligned}$$

0.11 Integration by parts

Another integration technique makes use of the product rule for differentiation. Since

$$(fg)' = f'g + fg',$$

we have

$$f'g = (fg)' - fg'.$$

Therefore,

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx.$$

Commonly, the above integral is done by writing

$$\begin{aligned} u &= g(x) & dv &= f'(x)dx \\ du &= g'(x)dx & v &= f(x). \end{aligned}$$

Then, the formula to be memorized is

$$\int udv = uv - \int vdu.$$

0.12 Taylor series

A Taylor series of a function $f(x)$ about a point $x = a$ is a power series representation of $f(x)$ developed so that all the derivatives of $f(x)$ at a match all the derivatives of the power series. Without worrying about convergence here, we have

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots.$$

Notice that the first term in the power series matches $f(a)$, all other terms vanishing, the second term matches $f'(a)$, all other terms vanishing, etc. Commonly, the Taylor

series is developed with $a = 0$. We will also make use of the Taylor series in a slightly different form, with $x = x_* + \epsilon$ and $a = x_*$:

$$f(x_* + \epsilon) = f(x_*) + f'(x_*)\epsilon + \frac{f''(x_*)}{2!}\epsilon^2 + \frac{f'''(x_*)}{3!}\epsilon^3 + \dots .$$

Another way to view this series is that of $g(\epsilon) = f(x_* + \epsilon)$, expanded about $\epsilon = 0$.

Taylor series that are commonly used include

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots, \\
\frac{1}{1+x} &= 1 - x + x^2 - \dots, \quad \text{for } |x| < 1, \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots, \quad \text{for } |x| < 1.
\end{aligned}$$

0.13 Functions of several variables

For simplicity, we consider a function $f = f(x, y)$ of two variables, though the results are easily generalized. The partial derivative of f with respect to x is defined as

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

and similarly for the partial derivative of f with respect to y. To take the partial derivative of f with respect to x, say, take the derivative of f with respect to x holding y fixed. As an example, consider

$$f(x, y) = 2x^3y^2 + y^3.$$

We have

$$\frac{\partial f}{\partial x} = 6x^2y^2, \quad \frac{\partial f}{\partial y} = 4x^3y + 3y^2.$$

Second derivatives are defined as the derivatives of the first derivatives, so we have

$$\frac{\partial^2 f}{\partial x^2} = 12xy^2, \quad \frac{\partial^2 f}{\partial y^2} = 4x^3 + 6y;$$

and the mixed second partial derivatives are

$$\frac{\partial^2 f}{\partial x \partial y} = 12x^2y, \quad \frac{\partial^2 f}{\partial y \partial x} = 12x^2y.$$

In general, mixed partial derivatives are independent of the order in which the derivatives are taken.

Partial derivatives are necessary for applying the chain rule. Consider

$$df = f(x + dx, y + dy) - f(x, y).$$

We can write df as

$$\begin{aligned} df &= [f(x+dx, y+dy) - f(x, y+dy)] + [f(x, y+dy) - f(x, y)] \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \end{aligned}$$

If one has $f = f(x(t), y(t))$, say, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

And if one has $f = f(x(r,\theta), y(r,\theta))$, say, then

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}.$$

A Taylor series of a function of several variables can also be developed. Here, all partial derivatives of $f(x, y)$ at (a, b) match all the partial derivatives of the power series. With the notation

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \quad \text{etc.,}$$

we have

$$\begin{aligned} f(x, y) &= f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \\ &+ \frac{1}{2!}\left(f_{xx}(a, b)(x-a)^2 + 2f_{xy}(a, b)(x-a)(y-b) + f_{yy}(a, b)(y-b)^2\right) + \dots \end{aligned}$$

0.14 Complex numbers

View tutorial on YouTube: Complex Numbers
View tutorial on YouTube: Complex Exponential Function

We define the imaginary number i to be one of the two numbers that satisfies the rule $(i)^2 = -1$, the other number being $-i$. Formally, we write $i = \sqrt{-1}$. A complex number z is written as

$$z = x + iy,$$

where x and y are real numbers. We call x the real part of z and y the imaginary part and write

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z.$$

Two complex numbers are equal if and only if their real and imaginary parts are equal.

The complex conjugate of $z = x + iy$, denoted as $\bar{z}$, is defined as

$$\bar{z} = x - iy.$$

Using z and $\bar{z}$, we have

$$\operatorname{Re} z = \frac{1}{2}(z + \bar{z}), \quad \operatorname{Im} z = \frac{1}{2i}(z - \bar{z}). \tag{4}$$

Furthermore,

$$\begin{aligned} z\bar{z} &= (x+iy)(x-iy) \\ &= x^2 - i^2y^2 \\ &= x^2 + y^2; \end{aligned}$$

and we define the absolute value of z, also called the modulus of z, by

$$\begin{aligned} |z| &= (z\bar{z})^{1/2} \\ &= \sqrt{x^2+y^2}. \end{aligned}$$

We can add, subtract, multiply and divide complex numbers to get new complex numbers. With $z = x + iy$ and $w = s + it$, and x, y, s, t real numbers, we have

$$z + w = (x+s) + i(y+t); \quad z - w = (x-s) + i(y-t);$$

$$\begin{aligned} zw &= (x+iy)(s+it) \\ &= (xs - yt) + i(xt + ys); \end{aligned}$$

$$\begin{aligned} \frac{z}{w} &= \frac{z\bar{w}}{w\bar{w}} \\ &= \frac{(x+iy)(s-it)}{s^2+t^2} \\ &= \frac{(xs+yt)}{s^2+t^2} + i\frac{(ys-xt)}{s^2+t^2}. \end{aligned}$$

Furthermore,

$$\begin{aligned} |zw| &= \sqrt{(xs-yt)^2 + (xt+ys)^2} \\ &= \sqrt{(x^2+y^2)(s^2+t^2)} \\ &= |z||w|; \end{aligned}$$

and

$$\begin{aligned} \overline{zw} &= (xs-yt) - i(xt+ys) \\ &= (x-iy)(s-it) \\ &= \bar{z}\bar{w}. \end{aligned}$$

Similarly

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}, \quad \overline{(\frac{z}{w})} = \frac{\bar{z}}{\bar{w}}.$$

Also, $\overline{z+w} = \bar{z} + \bar{w}$. However, $|z+w| \leq |z| + |w|$, a theorem known as the triangle inequality.

It is especially interesting and useful to consider the exponential function of an imaginary argument. Using the Taylor series expansion of an exponential function, we have

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right) \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

Since we have determined that

$$\cos\theta = \operatorname{Re} e^{i\theta}, \quad \sin\theta = \operatorname{Im} e^{i\theta}, \tag{5}$$

we also have using (4) and (5), the frequently used expressions

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

The much celebrated Euler's identity derives from $e^{i\theta} = \cos\theta + i\sin\theta$ by setting $\theta = \pi$, and using $\cos\pi = -1$ and $\sin\pi = 0$:

$$e^{i\pi} + 1 = 0,$$

and this identity links the five fundamental numbers, 0, 1, i, e and π, using three basic mathematical operations, addition, multiplication and exponentiation, only once.

The complex number z can be represented in the complex plane with $\operatorname{Re} z$ as the x-axis and $\operatorname{Im} z$ as the y-axis. This leads to the polar representation of $z = x + iy$:

$$z = re^{i\theta},$$

where $r = |z|$ and $\tan\theta = y/x$. We define $\arg z = \theta$. Note that θ is not unique, though it is conventional to choose the value such that $-\pi < \theta \leq \pi$, and $\theta = 0$ when $r = 0$.

Useful trigonometric relations can be derived using $e^{i\theta}$ and properties of the exponential function. The addition law can be derived from

$$e^{i(x+y)} = e^{ix}e^{iy}.$$

We have

$$\begin{aligned}\cos(x+y) + i\sin(x+y) &= (\cos x + i\sin x)(\cos y + i\sin y)\\ &= (\cos x\cos y - \sin x\sin y) + i(\sin x\cos y + \cos x\sin y);\end{aligned}$$

yielding

$$\cos(x+y) = \cos x\cos y - \sin x\sin y, \quad \sin(x+y) = \sin x\cos y + \cos x\sin y.$$

De Moivre's Theorem derives from $e^{in\theta} = (e^{i\theta})^n$, yielding the identity

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n.$$

For example, if $n = 2$, we derive

$$\begin{aligned}\cos 2\theta + i\sin 2\theta &= (\cos\theta + i\sin\theta)^2\\ &= (\cos^2\theta - \sin^2\theta) + 2i\cos\theta\sin\theta.\end{aligned}$$

Therefore,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\cos\theta\sin\theta.$$

With a little more manipulation using $\cos^2\theta + \sin^2\theta = 1$, we can derive

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2},$$

which are useful formulas for determining

$$\int \cos^2\theta\, d\theta = \frac{1}{4}(2\theta + \sin 2\theta) + c, \quad \int \sin^2\theta\, d\theta = \frac{1}{4}(2\theta - \sin 2\theta) + c,$$

from which follows

$$\int_0^{2\pi} \sin^2\theta\, d\theta = \int_0^{2\pi} \cos^2\theta\, d\theta = \pi.$$

Chapter 1

Introduction to odes

A differential equation is an equation for a function that relates the values of the function to the values of its derivatives. An ordinary differential equation (ode) is a differential equation for a function of a single variable, e.g., $x(t)$, while a partial differential equation (pde) is a differential equation for a function of several variables, e.g., $v(x, y, z, t)$. An ode contains ordinary derivatives and a pde contains partial derivatives. Typically, pde's are much harder to solve than ode's.

1.1 The simplest type of differential equation

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The simplest ordinary differential equations can be integrated directly by finding antiderivatives. These simplest odes have the form

$$\frac{d^n x}{dt^n} = G(t),$$

where the derivative of $x = x(t)$ can be of any order, and the right-hand-side may depend only on the independent variable t. As an example, consider a mass falling under the influence of constant gravity, such as approximately found on the Earth's surface. Newton's law, $F = ma$, results in the equation

$$m\frac{d^2x}{dt^2} = -mg,$$

where x is the height of the object above the ground, m is the mass of the object, and $g = 9.8$ meter/sec^2 is the constant gravitational acceleration. As Galileo suggested, the mass cancels from the equation, and

$$\frac{d^2x}{dt^2} = -g.$$

Here, the right-hand-side of the ode is a constant. The first integration, obtained by antidifferentiation, yields

$$\frac{dx}{dt} = A - gt,$$

with A the first constant of integration; and the second integration yields

$$x = B + At - \frac{1}{2}gt^2,$$

with B the second constant of integration. The two constants of integration A and B can then be determined from the initial conditions. If we know that the initial height of the mass is x_0, and the initial velocity is v_0, then the initial conditions are

$$x(0) = x_0, \quad \frac{dx}{dt}(0) = v_0.$$

Substitution of these initial conditions into the equations for dx/dt and x allows us to solve for A and B. The unique solution that satisfies both the ode and the initial conditions is given by

$$x(t) = x_0 + v_0 t - \frac{1}{2} g t^2. \tag{1.1}$$

For example, suppose we drop a ball off the top of a 50 meter building. How long will it take the ball to hit the ground? This question requires solution of (1.1) for the time T it takes for $x(T) = 0$, given $x_0 = 50$ meter and $v_0 = 0$. Solving for T,

$$\begin{aligned} T &= \sqrt{\frac{2x_0}{g}} \\ &= \sqrt{\frac{2 \cdot 50}{9.8}} \text{sec} \\ &\approx 3.2\text{sec}. \end{aligned}$$

Chapter 2

First-order differential equations

Reference: Boyce and DiPrima, Chapter 2

The general first-order differential equation for the function $y = y(x)$ is written as

$$\frac{dy}{dx} = f(x, y), \tag{2.1}$$

where $f(x, y)$ can be any function of the independent variable x and the dependent variable y. We first show how to determine a numerical solution of this equation, and then learn techniques for solving analytically some special forms of (2.1), namely, *separable* and *linear* first-order equations.

2.1 The Euler method

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Although it is not always possible to find an analytical solution of (2.1) for $y = y(x)$, it is always possible to determine a unique numerical solution given an initial value $y(x_0) = y_0$, and provided $f(x, y)$ is a well-behaved function. The differential equation (2.1) gives us the slope $f(x_0, y_0)$ of the tangent line to the solution curve $y = y(x)$ at the point (x_0, y_0). With a small step size $\Delta x = x_1 - x_0$, the initial condition (x_0, y_0) can be marched forward to (x_1, y_1) along the tangent line using Euler's method (see Fig. 2.1)

$$y_1 = y_0 + \Delta x f(x_0, y_0).$$

This solution (x_1, y_1) then becomes the new initial condition and is marched forward to (x_2, y_2) along a newly determined tangent line with slope given by $f(x_1, y_1)$. For small enough Δx, the numerical solution converges to the exact solution.

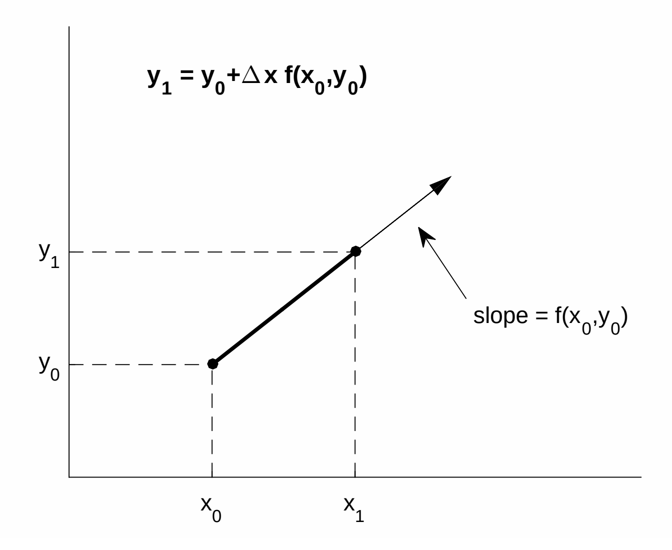


Figure 2.1: *The differential equation* $dy/dx = f(x, y)$, $y(x_0) = y_0$, *is integrated to* $x = x_1$ *using the Euler method* $y_1 = y_0 + \Delta x f(x_0, y_0)$, *with* $\Delta x = x_1 - x_0$.

2.2 Separable equations

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A first-order ode is separable if it can be written in the form

$$g(y)\frac{dy}{dx} = f(x), \quad y(x_0) = y_0, \tag{2.2}$$

where the function $g(y)$ is independent of x and $f(x)$ is independent of y. Integration from x_0 to x results in

$$\int_{x_0}^{x} g(y(x))y'(x)dx = \int_{x_0}^{x} f(x)dx.$$

The integral on the left can be transformed by substituting $u = y(x)$, $du = y'(x)dx$, and changing the lower and upper limits of integration to $y(x_0) = y_0$ and $y(x) = y$. Therefore,

$$\int_{y_0}^{y} g(u)du = \int_{x_0}^{x} f(x)dx,$$

and since u is a dummy variable of integration, we can write this in the equivalent form

$$\int_{y_0}^{y} g(y)dy = \int_{x_0}^{x} f(x)dx. \tag{2.3}$$

A simpler procedure that also yields (2.3) is to treat dy/dx in (2.2) like a fraction. Multiplying (2.2) by dx results in

$$g(y)dy = f(x)dx,$$

which is a separated equation with all the dependent variables on the left-side, and all the independent variables on the right-side. Equation (2.3) then results directly upon integration.

Example: Solve $\frac{dy}{dx} + \frac{1}{2}y = \frac{3}{2}$, with $y(0) = 2$.

We first manipulate the differential equation to the form

$$\frac{dy}{dx} = \frac{1}{2}(3 - y), \tag{2.4}$$

and then treat dy/dx as if it was a fraction to separate variables:

$$\frac{dy}{3 - y} = \frac{1}{2}dx.$$

We integrate the right-side from the initial condition $x = 0$ to x and the left-side from the initial condition $y(0) = 2$ to y. Accordingly,

$$\int_2^y \frac{dy}{3 - y} = \frac{1}{2}\int_0^x dx. \tag{2.5}$$

The integrals in (2.5) need to be done. Note that $y(x) < 3$ for finite x or the integral on the left-side diverges. Therefore, $3 - y > 0$ and integration yields

$$\begin{aligned} -\ln(3 - y)]_2^y &= \frac{1}{2}x]_0^x, \\ \ln(3 - y) &= -\frac{1}{2}x, \\ 3 - y &= e^{-\frac{1}{2}x}, \\ y &= 3 - e^{-\frac{1}{2}x}. \end{aligned}$$

Since this is our first nontrivial analytical solution, it is prudent to check our result. We do this by differentiating our solution:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}e^{-\frac{1}{2}x} \\ &= \frac{1}{2}(3 - y); \end{aligned}$$

and checking the initial conditions, $y(0) = 3 - e^0 = 2$. Therefore, our solution satisfies both the original ode and the initial condition.

Example: Solve $\frac{dy}{dx} + \frac{1}{2}y = \frac{3}{2}$, with $y(0) = 4$.

This is the identical differential equation as before, but with different initial conditions. We will jump directly to the integration step:

$$\int_4^y \frac{dy}{3 - y} = \frac{1}{2}\int_0^x dx.$$

Now $y(x) > 3$, so that $y - 3 > 0$ and integration yields

$$\begin{aligned} -\ln(y - 3)]_4^y &= \frac{1}{2}x]_0^x, \\ \ln(y - 3) &= -\frac{1}{2}x, \\ y - 3 &= e^{-\frac{1}{2}x}, \\ y &= 3 + e^{-\frac{1}{2}x}. \end{aligned}$$

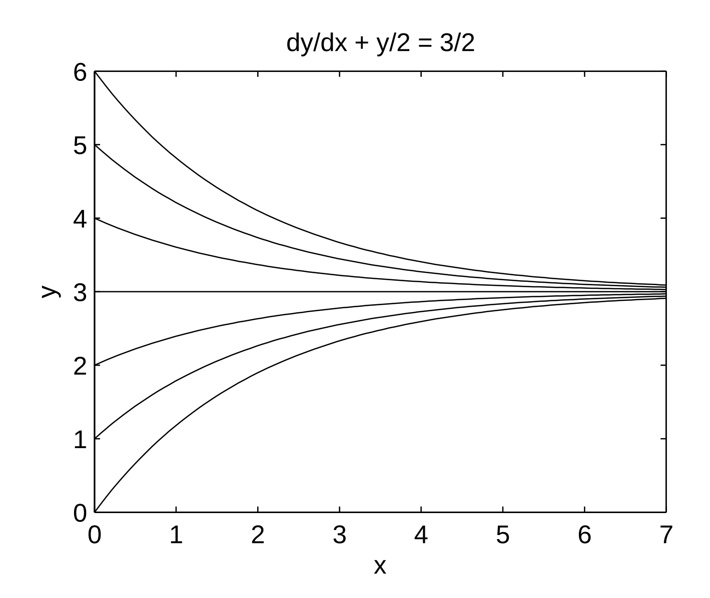


Figure 2.2: *Solution of the following ode:* $\frac{dy}{dx} + \frac{1}{2}y = \frac{3}{2}$.

The solution curves for a range of initial conditions is presented in Fig. 2.2. All solutions have a horizontal asymptote at $y = 3$ at which $dy/dx = 0$. For $y(0) = y_0$, the general solution can be shown to be $y(x) = 3 + (y_0 - 3)\exp(-x/2)$.

Example: Solve $\frac{dy}{dx} = \frac{2\cos 2x}{3+2y}$, with $y(0) = -1$. (i) For what values of $x > 0$ does the solution exist? (ii) For what value of $x > 0$ is $y(x)$ maximum?

Notice that the derivative of y diverges when $y = -3/2$, and that this may cause some problems with a solution.

We solve the ode by separating variables and integrating from initial conditions:

$$\begin{aligned} (3 + 2y)dy &= 2\cos 2x\, dx \\ \int_{-1}^{y} (3 + 2y)dy &= 2\int_{0}^{x} \cos 2x\, dx \\ 3y + y^2]_{-1}^{y} &= \sin 2x]_{0}^{x} \\ y^2 + 3y + 2 - \sin 2x &= 0 \\ y_\pm &= \frac{1}{2}[-3 \pm \sqrt{1 + 4\sin 2x}]. \end{aligned}$$

Solving the quadratic equation for y has introduced a spurious solution that does not satisfy the initial conditions. We test:

$$y_\pm(0) = \frac{1}{2}[-3 \pm 1] = \begin{cases} \text{-1;} \\ \text{-2.} \end{cases}$$

Only the $+$ root satisfies the initial condition, so that the unique solution to the ode and initial condition is

$$y = \frac{1}{2}[-3 + \sqrt{1 + 4\sin 2x}]. \tag{2.6}$$

To determine (i) the values of $x > 0$ for which the solution exists, we require

$$1 + 4\sin 2x \geq 0,$$

or

$$\sin 2x \geq -\frac{1}{4}. \tag{2.7}$$

Notice that at $x = 0$, we have $\sin 2x = 0$; at $x = \pi/4$, we have $\sin 2x = 1$; at $x = \pi/2$, we have $\sin 2x = 0$; and at $x = 3\pi/4$, we have $\sin 2x = -1$ We therefore need to determine the value of x such that $\sin 2x = -1/4$, with x in the range $\pi/2 < x < 3\pi/4$. The solution to the ode will then exist for all x between zero and this value.

To solve $\sin 2x = -1/4$ for x in the interval $\pi/2 < x < 3\pi/4$, one needs to recall the definition of arcsin, or $\sin^{-1}$, as found on a typical scientific calculator. The inverse of the function

$$f(x) = \sin x, \quad -\pi/2 \leq x \leq \pi/2$$

is denoted by arcsin. The first solution with $x > 0$ of the equation $\sin 2x = -1/4$ places $2x$ in the interval $(\pi, 3\pi/2)$, so to invert this equation using the arcsine we need to apply the identity $\sin(\pi - x) = \sin x$, and rewrite $\sin 2x = -1/4$ as $\sin(\pi - 2x) = -1/4$. The solution of this equation may then be found by taking the arcsine, and is

$$\pi - 2x = \arcsin(-1/4),$$

or

$$x = \frac{1}{2}\left(\pi + \arcsin\frac{1}{4}\right).$$

Therefore the solution exists for $0 \leq x \leq (\pi + \arcsin(1/4))/2 = 1.6971\ldots$, where we have used a calculator value (computing in radians) to find $\arcsin(0.25) = 0.2527\ldots$. At the value $(x, y) = (1.6971\ldots, -3/2)$, the solution curve ends and dy/dx becomes infinite.

To determine (ii) the value of x at which $y = y(x)$ is maximum, we examine (2.6) directly. The value of y will be maximum when $\sin 2x$ takes its maximum value over the interval where the solution exists. This will be when $2x = \pi/2$, or $x = \pi/4 = 0.7854\ldots$.

The graph of $y = y(x)$ is shown in Fig. 2.3.

2.3 Linear equations

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The linear first-order differential equation (linear in y and its derivative) can be written in the form

$$\frac{dy}{dx} + p(x)y = g(x), \tag{2.8}$$

with the initial condition $y(x_0) = y_0$. Linear first-order equations can be integrated using an integrating factor $\mu(x)$. We multiply (2.8) by $\mu(x)$,

$$\mu(x)\left[\frac{dy}{dx} + p(x)y\right] = \mu(x)g(x), \tag{2.9}$$

and try to determine $\mu(x)$ so that

$$\mu(x)\left[\frac{dy}{dx} + p(x)y\right] = \frac{d}{dx}[\mu(x)y]. \tag{2.10}$$

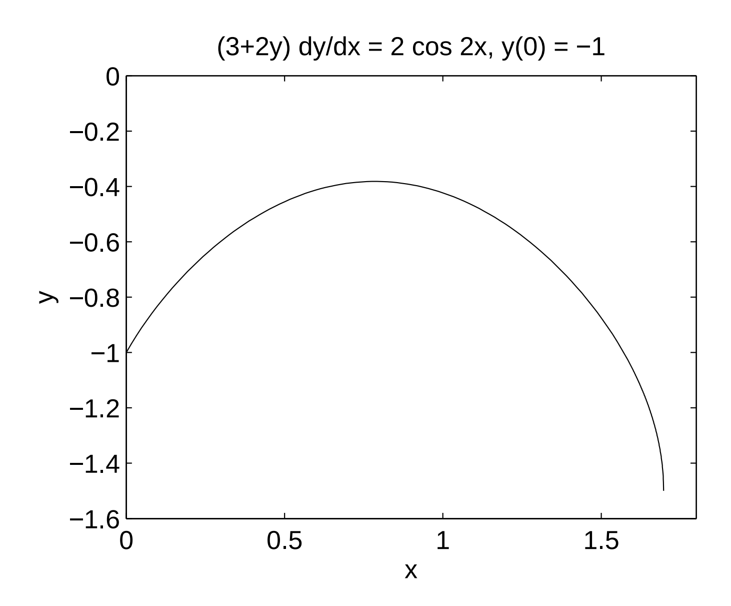


Figure 2.3: *Solution of the following ode:* $(3+2y)y' = 2\cos 2x$, $y(0) = -1$.

Equation (2.9) then becomes

$$\frac{d}{dx}[\mu(x)y] = \mu(x)g(x). \tag{2.11}$$

Equation (2.11) is easily integrated using $\mu(x_0) = \mu_0$ and $y(x_0) = y_0$:

$$\mu(x)y - \mu_0 y_0 = \int_{x_0}^{x} \mu(x)g(x)dx,$$

or

$$y = \frac{1}{\mu(x)}\left(\mu_0 y_0 + \int_{x_0}^{x} \mu(x)g(x)dx\right). \tag{2.12}$$

It remains to determine $\mu(x)$ from (2.10). Differentiating and expanding (2.10) yields

$$\mu\frac{dy}{dx} + p\mu y = \frac{d\mu}{dx}y + \mu\frac{dy}{dx};$$

and upon simplifying,

$$\frac{d\mu}{dx} = p\mu. \tag{2.13}$$

Equation (2.13) is separable and can be integrated:

$$\begin{aligned}\int_{\mu_0}^{\mu} \frac{d\mu}{\mu} &= \int_{x_0}^{x} p(x)dx,\\ \ln\frac{\mu}{\mu_0} &= \int_{x_0}^{x} p(x)dx,\\ \mu(x) &= \mu_0 \exp\left(\int_{x_0}^{x} p(x)dx\right).\end{aligned}$$

Notice that since μ_0 cancels out of (2.12), it is customary to assign $\mu_0 = 1$. The solution to (2.8) satisfying the initial condition $y(x_0) = y_0$ is then commonly written

as

$$y = \frac{1}{\mu(x)} \left(y_0 + \int_{x_0}^{x} \mu(x) g(x) dx \right),$$

with

$$\mu(x) = \exp \left(\int_{x_0}^{x} p(x) dx \right)$$

the integrating factor. This important result finds frequent use in applied mathematics.

Example: Solve $\frac{dy}{dx} + 2y = e^{-x}$, with $y(0) = 3/4$.

Note that this equation is not separable. With $p(x) = 2$ and $g(x) = e^{-x}$, we have

$$\begin{aligned} \mu(x) &= \exp \left(\int_0^x 2dx \right) \\ &= e^{2x}, \end{aligned}$$

and

$$\begin{aligned} y &= e^{-2x} \left(\frac{3}{4} + \int_0^x e^{2x} e^{-x} dx \right) \\ &= e^{-2x} \left(\frac{3}{4} + \int_0^x e^{x} dx \right) \\ &= e^{-2x} \left(\frac{3}{4} + (e^x - 1) \right) \\ &= e^{-2x} \left(e^x - \frac{1}{4} \right) \\ &= e^{-x} \left(1 - \frac{1}{4} e^{-x} \right). \end{aligned}$$

Example: Solve $\frac{dy}{dx} - 2xy = x$, with $y(0) = 0$.

This equation is separable, and we solve it in two ways. First, using an integrating factor with $p(x) = -2x$ and $g(x) = x$:

$$\begin{aligned} \mu(x) &= \exp \left(-2 \int_0^x x dx \right) \\ &= e^{-x^2}, \end{aligned}$$

and

$$y = e^{x^2} \int_0^x x e^{-x^2} dx.$$

The integral can be done by substitution with $u = x^2$, $du = 2xdx$:

$$\begin{aligned} \int_0^x x e^{-x^2} dx &= \frac{1}{2} \int_0^{x^2} e^{-u} du \\ &= -\frac{1}{2} e^{-u} \Big]_0^{x^2} \\ &= \frac{1}{2} \left(1 - e^{-x^2} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} y &= \frac{1}{2} e^{x^2} \left(1 - e^{-x^2}\right) \\ &= \frac{1}{2} \left(e^{x^2} - 1\right). \end{aligned}$$

Second, we integrate by separating variables:

$$\begin{aligned} \frac{dy}{dx} - 2xy &= x, \\ \frac{dy}{dx} &= x(1+2y), \\ \int_0^y \frac{dy}{1+2y} &= \int_0^x x dx, \\ \frac{1}{2} \ln(1+2y) &= \frac{1}{2} x^2, \\ 1 + 2y &= e^{x^2}, \\ y &= \frac{1}{2} \left(e^{x^2} - 1\right). \end{aligned}$$

The results from the two different solution methods are the same, and the choice of method is a personal preference.

2.4 Applications

2.4.1 Compound interest

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The equation for the growth of an investment with continuous compounding of interest is a first-order differential equation. Let $S(t)$ be the value of the investment at time t, and let r be the annual interest rate compounded after every time interval Δt. We can also include deposits (or withdrawals). Let k be the annual deposit amount, and suppose that an installment is deposited after every time interval Δt. The value of the investment at the time $t + \Delta t$ is then given by

$$S(t + \Delta t) = S(t) + (r\Delta t)S(t) + k\Delta t, \tag{2.14}$$

where at the end of the time interval Δt, $r\Delta t S(t)$ is the amount of interest credited and $k\Delta t$ is the amount of money deposited ($k > 0$) or withdrawn ($k < 0$). As a numerical example, if the account held \$10,000 at time t, and $r = 6\%$ per year and $k = \$12{,}000$ per year, say, and the compounding and deposit period is $\Delta t = 1$ month $= 1/12$ year, then the interest awarded after one month is $r\Delta t S = (0.06/12) \times \$10{,}000 = \$50$, and the amount deposited is $k\Delta t = \$1000$.

Rearranging the terms of (2.14) to exhibit what will soon become a derivative, we have

$$\frac{S(t + \Delta t) - S(t)}{\Delta t} = rS(t) + k.$$

The equation for continuous compounding of interest and continuous deposits is obtained by taking the limit $\Delta t \to 0$. The resulting differential equation is

$$\frac{dS}{dt} = rS + k, \tag{2.15}$$

which can solved with the initial condition $S(0) = S_0$, where S_0 is the initial capital. We can solve either by separating variables or by using an integrating factor; I solve here by separating variables. Integrating from $t = 0$ to a final time t,

$$\begin{aligned}
\int_{S_0}^{S} \frac{dS}{rS + k} &= \int_0^t dt, \\
\frac{1}{r} \ln\left(\frac{rS + k}{rS_0 + k}\right) &= t, \\
rS + k &= (rS_0 + k)e^{rt}, \\
S &= \frac{rS_0 e^{rt} + ke^{rt} - k}{r}, \\
S &= S_0 e^{rt} + \frac{k}{r} e^{rt}\left(1 - e^{-rt}\right), \qquad (2.16)
\end{aligned}$$

where the first term on the right-hand side of (2.16) comes from the initial invested capital, and the second term comes from the deposits (or withdrawals). Evidently, compounding results in the exponential growth of an investment.

As a practical example, we can analyze a simple retirement plan. It is easiest to assume that all amounts and returns are in real dollars (adjusted for inflation). Suppose a 25 year-old plans to set aside a fixed amount every year of his/her working life, invests at a real return of 6%, and retires at age 65. How much must he/she invest each year to have \$8,000,000 at retirement? We need to solve (2.16) for k using $t = 40$ years, $S(t) = \$8{,}000{,}000$, $S_0 = 0$, and $r = 0.06$ per year. We have

$$\begin{aligned}
k &= \frac{rS(t)}{e^{rt} - 1}, \\
k &= \frac{0.06 \times 8{,}000{,}000}{e^{0.06 \times 40} - 1}, \\
&= \$47{,}889 \text{ year}^{-1}.
\end{aligned}$$

To have saved approximately one million US\$ at retirement, the worker would need to save about HK\$50,000 per year over his/her working life. Note that the amount saved over the worker's life is approximately $40 \times \$50{,}000 = \$2{,}000{,}000$, while the amount earned on the investment (at the assumed 6% real return) is approximately $\$8{,}000{,}000 - \$2{,}000{,}000 = \$6{,}000{,}000$. The amount earned from the investment is about $3\times$ the amount saved, even with the modest real return of 6%. Sound investment planning is well worth the effort.

2.4.2 Chemical reactions

Suppose that two chemicals A and B react to form a product C, which we write as

$$A + B \xrightarrow{k} C,$$

where k is called the rate constant of the reaction. For simplicity, we will use the same symbol C, say, to refer to both the chemical C and its concentration. The law of mass action says that dC/dt is proportional to the product of the concentrations A and B, with proportionality constant k; that is,

$$\frac{dC}{dt} = kAB. \qquad (2.17)$$

Similarly, the law of mass action enables us to write equations for the time-derivatives of the reactant concentrations A and B:

$$\frac{dA}{dt} = -kAB, \quad \frac{dB}{dt} = -kAB. \tag{2.18}$$

The ode given by (2.17) can be solved analytically using conservation laws. We assume that A_0 and B_0 are the initial concentrations of the reactants, and that no product is initially present. From (2.17) and (2.18),

$$\begin{aligned} \frac{d}{dt}(A+C) &= 0 \qquad \Longrightarrow \quad A + C = A_0, \\ \frac{d}{dt}(B+C) &= 0 \qquad \Longrightarrow \quad B + C = B_0. \end{aligned}$$

Using these conservation laws, (2.17) becomes

$$\frac{dC}{dt} = k(A_0 - C)(B_0 - C), \quad C(0) = 0,$$

which is a nonlinear equation that may be integrated by separating variables. Separating and integrating, we obtain

$$\begin{aligned} \int_0^C \frac{dC}{(A_0 - C)(B_0 - C)} &= k \int_0^t dt \\ &= kt. \end{aligned} \tag{2.19}$$

The remaining integral can be done using the method of partial fractions. We write

$$\frac{1}{(A_0 - C)(B_0 - C)} = \frac{a}{A_0 - C} + \frac{b}{B_0 - C}. \tag{2.20}$$

The cover-up method is the simplest method to determine the unknown coefficients a and b. To determine a, we multiply both sides of (2.20) by $A_0 - C$ and set $C = A_0$ to find

$$a = \frac{1}{B_0 - A_0}.$$

Similarly, to determine b, we multiply both sides of (2.20) by $B_0 - C$ and set $C = B_0$ to find

$$b = \frac{1}{A_0 - B_0}.$$

Therefore,

$$\frac{1}{(A_0 - C)(B_0 - C)} = \frac{1}{B_0 - A_0}\left(\frac{1}{A_0 - C} - \frac{1}{B_0 - C}\right),$$

and the remaining integral of (2.19) becomes (using $C < A_0, B_0$)

$$\begin{aligned} \int_0^C \frac{dC}{(A_0 - C)(B_0 - C)} &= \frac{1}{B_0 - A_0}\left(\int_0^C \frac{dC}{A_0 - C} - \int_0^C \frac{dC}{B_0 - C}\right) \\ &= \frac{1}{B_0 - A_0}\left(-\ln\left(\frac{A_0 - C}{A_0}\right) + \ln\left(\frac{B_0 - C}{B_0}\right)\right) \\ &= \frac{1}{B_0 - A_0}\ln\left(\frac{A_0(B_0 - C)}{B_0(A_0 - C)}\right). \end{aligned}$$

Using this integral in (2.19), multiplying by $(B_0 - A_0)$ and exponentiating, we obtain

$$\frac{A_0(B_0 - C)}{B_0(A_0 - C)} = e^{(B_0 - A_0)kt}.$$

Solving for C, we finally obtain

$$C(t) = A_0 B_0 \frac{e^{(B_0-A_0)kt} - 1}{B_0 e^{(B_0-A_0)kt} - A_0},$$

which appears to be a complicated expression, but has the simple limits

$$\begin{aligned}\lim_{t\to\infty} C(t) &= \begin{cases} A_0, & \text{if } A_0 < B_0, \\ B_0, & \text{if } B_0 < A_0 \end{cases} \\ &= \min(A_0, B_0).\end{aligned}$$

As one would expect, the reaction stops after one of the reactants is depleted; and the final concentration of product is equal to the initial concentration of the depleted reactant.

2.4.3 Terminal velocity

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Using Newton's law, we model a mass m free falling under gravity but with air resistance. We assume that the force of air resistance is proportional to the speed of the mass and opposes the direction of motion. We define the x-axis to point in the upward direction, opposite the force of gravity. Near the surface of the Earth, the force of gravity is approximately constant and is given by $-mg$, with $g = 9.8\,\text{m/s}^2$ the usual gravitational acceleration. The force of air resistance is modeled by $-kv$, where v is the vertical velocity of the mass and k is a positive constant. When the mass is falling, $v < 0$ and the force of air resistance is positive, pointing upward and opposing the motion. The total force on the mass is therefore given by $F = -mg - kv$. With $F = ma$ and $a = dv/dt$, we obtain the differential equation

$$m\frac{dv}{dt} = -mg - kv. \tag{2.21}$$

The terminal velocity v_∞ of the mass is defined as the asymptotic velocity after air resistance balances the gravitational force. When the mass is at terminal velocity, $dv/dt = 0$ so that

$$v_\infty = -\frac{mg}{k}. \tag{2.22}$$

The approach to the terminal velocity of a mass initially at rest is obtained by solving (2.21) with initial condition $v(0) = 0$. The equation is both linear and separable, and I solve by separating variables:

$$\begin{aligned} m\int_0^v \frac{dv}{mg + kv} &= -\int_0^t dt, \\ \frac{m}{k}\ln\left(\frac{mg + kv}{mg}\right) &= -t, \\ 1 + \frac{kv}{mg} &= e^{-kt/m}, \\ v &= -\frac{mg}{k}\left(1 - e^{-kt/m}\right). \end{aligned}$$

Therefore, $v = v_\infty \left(1 - e^{-kt/m}\right)$, and v approaches v_∞ as the exponential term decays to zero.

As an example, a skydiver of mass $m = 100\,\text{kg}$ with his parachute closed may have a terminal velocity of $200\,\text{km/hr}$. With

$$g = (9.8\,\text{m/s}^2)(10^{-3}\,\text{km/m})(60\,\text{s/min})^2(60\,\text{min/hr})^2 = 127,008\,\text{km/hr}^2,$$

one obtains from (2.22), $k = 63,504\,\text{kg/hr}$. One-half of the terminal velocity for free-fall ($100\,\text{km/hr}$) is therefore attained when $(1 - e^{-kt/m}) = 1/2$, or $t = m\ln 2/k \approx 4\,\text{sec}$. Approximately 95% of the terminal velocity ($190\,\text{km/hr}$) is attained after $17\,\text{sec}$.

2.4.4 Escape velocity

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An interesting physical problem is to find the smallest initial velocity for a mass on the Earth's surface to escape from the Earth's gravitational field, the so-called escape velocity. Newton's law of universal gravitation asserts that the gravitational force between two massive bodies is proportional to the product of the two masses and inversely proportional to the square of the distance between them. For a mass m a position x above the surface of the Earth, the force on the mass is given by

$$F = -G\frac{Mm}{(R+x)^2},$$

where M and R are the mass and radius of the Earth and G is the gravitational constant. The minus sign means the force on the mass m points in the direction of decreasing x. The approximately constant acceleration g on the Earth's surface corresponds to the absolute value of F/m when $x = 0$:

$$g = \frac{GM}{R^2},$$

and $g \approx 9.8\ \text{m/s}^2$. Newton's law $F = ma$ for the mass m is thus given by

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{GM}{(R+x)^2} \\ &= -\frac{g}{(1+x/R)^2}, \end{aligned} \tag{2.23}$$

where the radius of the Earth is known to be $R \approx 6350\,\text{km}$.

A useful trick allows us to solve this second-order differential equation as a first-order equation. First, note that $d^2x/dt^2 = dv/dt$. If we write $v(t) = v(x(t))$—considering the velocity of the mass m to be a function of its distance above the Earth—we have using the chain rule

$$\begin{aligned} \frac{dv}{dt} &= \frac{dv}{dx}\frac{dx}{dt} \\ &= v\frac{dv}{dx}, \end{aligned}$$

where we have used $v = dx/dt$. Therefore, (2.23) becomes the first-order ode

$$v\frac{dv}{dx} = -\frac{g}{(1+x/R)^2},$$

which may be solved assuming an initial velocity $v(x = 0) = v_0$ when the mass is shot vertically from the Earth's surface. Separating variables and integrating, we obtain

$$\int_{v_0}^{v} v\,dv = -g \int_0^x \frac{dx}{(1 + x/R)^2}.$$

The left integral is $\frac{1}{2}(v^2 - v_0^2)$, and the right integral can be performed using the substitution $u = 1 + x/R$, $du = dx/R$:

$$\begin{aligned}
\int_0^x \frac{dx}{(1 + x/R)^2} &= R \int_1^{1+x/R} \frac{du}{u^2} \\
&= -\frac{R}{u}\bigg]_1^{1+x/R} \\
&= R - \frac{R^2}{x + R} \\
&= \frac{Rx}{x + R}.
\end{aligned}$$

Therefore,

$$\frac{1}{2}(v^2 - v_0^2) = -\frac{gRx}{x + R},$$

which when multiplied by m is an expression of the conservation of energy (the change of the kinetic energy of the mass is equal to the change in the potential energy). Solving for v^2,

$$v^2 = v_0^2 - \frac{2gRx}{x + R}.$$

The escape velocity is defined as the minimum initial velocity v_0 such that the mass can *escape* to infinity. Therefore, $v_0 = v_{\text{escape}}$ when $v \to 0$ as $x \to \infty$. Taking this limit, we have

$$\begin{aligned}
v_{\text{escape}}^2 &= \lim_{x \to \infty} \frac{2gRx}{x + R} \\
&= 2gR.
\end{aligned}$$

With $R \approx 6350$ km and $g = 127\,008\,\text{km/hr}^2$, we determine $v_{\text{escape}} = \sqrt{2gR} \approx 40\,000$ km/hr. In comparison, the muzzle velocity of a modern high-performance rifle is 4300 km/hr, almost an order of magnitude too slow for a bullet, shot into the sky, to escape the Earth's gravity.

2.4.5 RC circuit

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Consider a resister R and a capacitor C connected in series as shown in Fig. 2.4. A battery providing an electromotive force, or emf $\mathcal{E}$, connects to this circuit by a switch. Initially, there is no charge on the capacitor. When the switch is thrown to a, the battery connects and the capacitor charges. When the switch is thrown to b, the battery disconnects and the capacitor discharges, with energy dissipated in the resister. Here, we determine the voltage drop across the capacitor during charging and discharging.

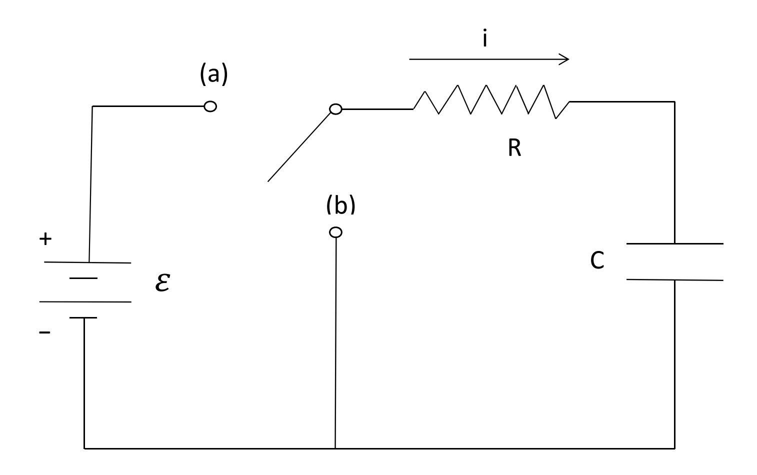


Figure 2.4: RC circuit diagram.

The equations for the voltage drops across a capacitor and a resister are given by

$$V_C = q/C, \quad V_R = iR, \tag{2.24}$$

where C is the capacitance and R is the resistance. The charge q and the current i are related by

$$i = \frac{dq}{dt}. \tag{2.25}$$

Kirchhoff's voltage law states that the emf $\mathcal{E}$ in any closed loop is equal to the sum of the voltage drops in that loop. Applying Kirchhoff's voltage law when the switch is thrown to a results in

$$V_R + V_C = \mathcal{E}. \tag{2.26}$$

Using (2.24) and (2.25), the voltage drop across the resister can be written in terms of the voltage drop across the capacitor as

$$V_R = RC\frac{dV_C}{dt},$$

and (2.26) can be rewritten to yield the linear first-order differential equation for V_C given by

$$\frac{dV_C}{dt} + V_C/RC = \mathcal{E}/RC, \tag{2.27}$$

with initial condition $V_C(0) = 0$.

The integrating factor for this equation is

$$\mu(t) = e^{t/RC},$$

and (2.27) integrates to

$$V_C(t) = e^{-t/RC}\int_0^t (\mathcal{E}/RC)e^{t/RC}dt,$$

with solution

$$V_C(t) = \mathcal{E}\left(1 - e^{-t/RC}\right).$$

The voltage starts at zero and rises exponentially to $\mathcal{E}$, with characteristic time scale given by RC.

When the switch is thrown to b, application of Kirchhoff's voltage law results in

$$V_R + V_C = 0,$$

with corresponding differential equation

$$\frac{dV_C}{dt} + V_C/RC = 0.$$

Here, we assume that the capacitance is initially fully charged so that $V_C(0) = \mathcal{E}$. The solution, then, during the discharge phase is given by

$$V_C(t) = \mathcal{E}e^{-t/RC}.$$

The voltage starts at $\mathcal{E}$ and decays exponentially to zero, again with characteristic time scale given by RC.

2.4.6 The logistic equation

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Let $N(t)$ be the number of individuals in a population at time t, and let b and d be the average per capita birth rate and death rate, respectively. In a short time Δt, the number of births in the population is $b\Delta t N$, and the number of deaths is $d\Delta t N$. An equation for N at time $t + \Delta t$ is then determined to be

$$N(t + \Delta t) = N(t) + b\Delta t N(t) - d\Delta t N(t),$$

which can be rearranged to

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} = (b - d)N(t);$$

and as $\Delta t \to 0$, and with $r = b - d$, we have

$$\frac{dN}{dt} = rN.$$

This is the Malthusian growth model (Thomas Malthus, 1766-1834), and is the same equation as our compound interest model.

Under a Malthusian growth model, the population size grows exponentially like

$$N(t) = N_0 e^{rt},$$

where N_0 is the initial population size. However, when the population growth is constrained by limited resources, a heuristic modification to the Malthusian growth model results in the Verhulst equation,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \tag{2.28}$$

where K is called the carrying capacity of the environment. Making (2.28) dimensionless using $\tau = rt$ and $x = N/K$ leads to the logistic equation,

$$\frac{dx}{d\tau} = x(1 - x),$$

where we may assume the initial condition $x(0) = x_0 > 0$. Separating variables and integrating

$$\int_{x_0}^{x} \frac{dx}{x(1-x)} = \int_0^{\tau} d\tau.$$

The integral on the left-hand-side can be done using the method of partial fractions:

$$\frac{1}{x(1-x)} = \frac{a}{x} + \frac{b}{1-x},$$

and the cover-up method yields $a = b = 1$. Therefore,

$$\begin{aligned}
\int_{x_0}^{x} \frac{dx}{x(1-x)} &= \int_{x_0}^{x} \frac{dx}{x} + \int_{x_0}^{x} \frac{dx}{(1-x)} \\
&= \ln \frac{x}{x_0} - \ln \frac{1-x}{1-x_0} \\
&= \ln \frac{x(1-x_0)}{x_0(1-x)} \\
&= \tau.
\end{aligned}$$

Solving for x, we first exponentiate both sides and then isolate x:

$$\begin{aligned}
\frac{x(1-x_0)}{x_0(1-x)} &= e^{\tau}, \\
x(1-x_0) &= x_0 e^{\tau} - x x_0 e^{\tau}, \\
x(1 - x_0 + x_0 e^{\tau}) &= x_0 e^{\tau}, \\
x &= \frac{x_0}{x_0 + (1-x_0)e^{-\tau}}. \qquad (2.29)
\end{aligned}$$

We observe that for $x_0 > 0$, we have $\lim_{\tau \to \infty} x(\tau) = 1$, corresponding to

$$\lim_{t \to \infty} N(t) = K.$$

The population, therefore, grows in size until it reaches the carrying capacity of its environment.

Chapter 3

Linear second-order differential equations with constant coefficients

Reference: Boyce and DiPrima, Chapter 3

The general linear second-order differential equation with independent variable t and dependent variable $x = x(t)$ is given by

$$\ddot{x} + p(t)\dot{x} + q(t)x = g(t), \tag{3.1}$$

where we have used the standard physics notation $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$. A unique solution of (3.1) requires initial values $x(t_0) = x_0$ and $\dot{x}(t_0) = u_0$. The equation with constant coefficients—on which we will devote considerable effort—assumes that $p(t)$ and $q(t)$ are constants, independent of time. The second-order linear ode is said to be *homogeneous* if $g(t) = 0$.

3.1 The Euler method

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In general, (3.1) cannot be solved analytically, and we begin by deriving an algorithm for numerical solution. Consider the general second-order ode given by

$$\ddot{x} = f(t, x, \dot{x}).$$

We can write this second-order ode as a pair of first-order odes by defining $u = \dot{x}$, and writing the first-order system as

$$\dot{x} = u, \tag{3.2}$$

$$\dot{u} = f(t, x, u). \tag{3.3}$$

The first ode, (3.2), gives the slope of the tangent line to the curve $x = x(t)$; the second ode, (3.3), gives the slope of the tangent line to the curve $u = u(t)$. Beginning at the initial values $(x, u) = (x_0, u_0)$ at the time $t = t_0$, we move along the tangent lines to determine $x_1 = x(t_0 + \Delta t)$ and $u_1 = u(t_0 + \Delta t)$:

$$x_1 = x_0 + \Delta t u_0,$$
$$u_1 = u_0 + \Delta t f(t_0, x_0, u_0).$$

The values x_1 and u_1 at the time $t_1 = t_0 + \Delta t$ are then used as new initial values to march the solution forward to time $t_2 = t_1 + \Delta t$. As long as $f(t, x, u)$ is a well-behaved function, the numerical solution converges to the unique solution of the ode as $\Delta t \to 0$.

3.2 The principle of superposition

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Consider the linear second-order homogeneous ode:

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0; \tag{3.4}$$

and suppose that $x = X_1(t)$ and $x = X_2(t)$ are solutions to (3.4). We consider a linear combination of X_1 and X_2 by letting

$$X(t) = c_1 X_1(t) + c_2 X_2(t), \tag{3.5}$$

with c_1 and c_2 constants. The *principle of superposition* states that $x = X(t)$ is also a solution of (3.4). To prove this, we compute

$$\begin{aligned} \ddot{X} + p\dot{X} + qX &= c_1\ddot{X}_1 + c_2\ddot{X}_2 + p\left(c_1\dot{X}_1 + c_2\dot{X}_2\right) + q\left(c_1 X_1 + c_2 X_2\right) \\ &= c_1\left(\ddot{X}_1 + p\dot{X}_1 + qX_1\right) + c_2\left(\ddot{X}_2 + p\dot{X}_2 + qX_2\right) \\ &= c_1 \times 0 + c_2 \times 0 \\ &= 0, \end{aligned}$$

since X_1 and X_2 were assumed to be solutions of (3.4). We have therefore shown that any linear combination of solutions to the homogeneous linear second-order ode is also a solution.

3.3 The Wronskian

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Suppose that having determined that two solutions of (3.4) are $x = X_1(t)$ and $x = X_2(t)$, we attempt to write the general solution to (3.4) as (3.5). We must then ask whether this general solution will be able to satisfy the two initial conditions given by

$$x(t_0) = x_0, \quad \dot{x}(t_0) = u_0. \tag{3.6}$$

Applying these initial conditions to (3.5), we obtain

$$\begin{aligned} c_1 X_1(t_0) + c_2 X_2(t_0) &= x_0, \\ c_1 \dot{X}_1(t_0) + c_2 \dot{X}_2(t_0) &= u_0, \end{aligned} \tag{3.7}$$

which is observed to be a system of two linear equations for the two unknowns c_1 and c_2. Solution of (3.7) by standard methods results in

$$c_1 = \frac{x_0\dot{X}_2(t_0) - u_0 X_2(t_0)}{W}, \quad c_2 = \frac{u_0 X_1(t_0) - x_0\dot{X}_1(t_0)}{W},$$

where W is called the Wronskian and is given by

$$W = X_1(t_0)\dot{X}_2(t_0) - \dot{X}_1(t_0)X_2(t_0). \tag{3.8}$$

Evidently, the Wronskian must not be equal to zero ($W \neq 0$) for a solution to exist.

For examples, the two solutions

$$X_1(t) = A\sin\omega t, \quad X_2(t) = B\sin\omega t,$$

have a zero Wronskian at $t = t_0$, as can be shown by computing

$$\begin{aligned} W &= (A \sin \omega t_0)(B\omega \cos \omega t_0) - (A\omega \cos \omega t_0)(B \sin \omega t_0) \\ &= 0; \end{aligned}$$

while the two solutions

$$X_1(t) = \sin \omega t, \quad X_2(t) = \cos \omega t,$$

with $\omega \neq 0$, have a nonzero Wronskian at $t = t_0$,

$$\begin{aligned} W &= (\sin \omega t_0)(-\omega \sin \omega t_0) - (\omega \cos \omega t_0)(\cos \omega t_0) \\ &= -\omega. \end{aligned}$$

When the Wronskian is not equal to zero, we say that the two solutions $X_1(t)$ and $X_2(t)$ are linearly independent. The concept of linear independence is borrowed from linear algebra, and indeed, the set of all functions that satisfy (3.4) can be shown to form a two-dimensional vector space.

3.4 Homogeneous linear second-order ode with constant coefficients

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We now study solutions of the homogeneous, constant coefficient ode, written as

$$a\ddot{x} + b\dot{x} + cx = 0, \tag{3.9}$$

with a, b, and c constants. Such an equation arises for the charge on a capacitor in an unpowered RLC electrical circuit, or for the position of a freely-oscillating frictional mass on a spring, or for a damped pendulum. Our solution method finds two linearly independent solutions to (3.9), multiplies each of these solutions by a constant, and adds them. The two free constants can then be used to satisfy two given initial conditions.

Because of the differential properties of the exponential function, a natural ansatz, or educated guess, for the form of the solution to (3.9) is $x = e^{rt}$, where r is a constant to be determined. Successive differentiation results in $\dot{x} = re^{rt}$ and $\ddot{x} = r^2e^{rt}$, and substitution into (3.9) yields

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0. \tag{3.10}$$

Our choice of exponential function is now rewarded by the explicit cancelation in (3.10) of e^{rt}. The result is a quadratic equation for the unknown constant r:

$$ar^2 + br + c = 0. \tag{3.11}$$

Our ansatz has thus converted a differential equation into an algebraic equation. Equation (3.11) is called the *characteristic equation* of (3.9). Using the quadratic formula, the two solutions of the characteristic equation (3.11) are given by

$$r_\pm = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right).$$

There are three cases to consider: (1) if $b^2 - 4ac > 0$, then the two roots are distinct and real; (2) if $b^2 - 4ac < 0$, then the two roots are distinct and complex conjugates of each other; (3) if $b^2 - 4ac = 0$, then the two roots are degenerate and there is only one real root. We will consider these three cases in turn.

3.4.1 Distinct real roots

When $r_+ \neq r_-$ are real roots, then the general solution to (3.9) can be written as a linear superposition of the two solutions $e^{r_+ t}$ and $e^{r_- t}$; that is,

$$x(t) = c_1 e^{r_+ t} + c_2 e^{r_- t}.$$

The unknown constants c_1 and c_2 can then be determined by the given initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = u_0$. We now present two examples.

Example 1: Solve $\ddot{x} + 5\dot{x} + 6x = 0$ with $x(0) = 2$, $\dot{x}(0) = 3$, and find the maximum value attained by x.

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We take as our ansatz $x = e^{rt}$ and obtain the characteristic equation

$$r^2 + 5r + 6 = 0,$$

which factors to

$$(r+3)(r+2) = 0.$$

The general solution to the ode is thus

$$x(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

The solution for $\dot{x}$ obtained by differentiation is

$$\dot{x}(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}.$$

Use of the initial conditions then results in two equations for the two unknown constant c_1 and c_2:

$$\begin{aligned} c_1 + c_2 &= 2, \\ -2c_1 - 3c_2 &= 3. \end{aligned}$$

Adding three times the first equation to the second equation yields $c_1 = 9$; and the first equation then yields $c_2 = 2 - c_1 = -7$. Therefore, the unique solution that satisfies both the ode and the initial conditions is

$$\begin{aligned} x(t) &= 9e^{-2t} - 7e^{-3t} \\ &= 9e^{-2t}\left(1 - \frac{7}{9}e^{-t}\right). \end{aligned}$$

Note that although both exponential terms decay in time, their sum increases initially since $\dot{x}(0) > 0$. The maximum value of x occurs at the time t_m when $\dot{x} = 0$, or

$$t_m = \ln(7/6).$$

The maximum $x_m = x(t_m)$ is then determined to be

$$x_m = 108/49.$$

Example 2: Solve $\ddot{x} - x = 0$ with $x(0) = x_0$, $\dot{x}(0) = u_0$.

Again our ansatz is $x = e^{rt}$, and we obtain the characteristic equation

$$r^2 - 1 = 0,$$

with solution $r_\pm = \pm 1$. Therefore, the general solution for x is

$$x(t) = c_1 e^t + c_2 e^{-t},$$

and the derivative satisfies

$$\dot{x}(t) = c_1 e^t - c_2 e^{-t}.$$

Initial conditions are satisfied when

$$\begin{aligned} c_1 + c_2 &= x_0, \\ c_1 - c_2 &= u_0. \end{aligned}$$

Adding and subtracting these equations, we determine

$$c_1 = \frac{1}{2}\left(x_0 + u_0\right), \quad c_2 = \frac{1}{2}\left(x_0 - u_0\right),$$

so that after rearranging terms

$$x(t) = x_0\left(\frac{e^t + e^{-t}}{2}\right) + u_0\left(\frac{e^t - e^{-t}}{2}\right).$$

The terms in parentheses are the usual definitions of the hyperbolic cosine and sine functions; that is,

$$\cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2}.$$

Our solution can therefore be rewritten as

$$x(t) = x_0 \cosh t + u_0 \sinh t.$$

Note that the relationships between the trigonometric functions and the complex exponentials were given by

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i},$$

so that

$$\cosh t = \cos it, \quad \sinh t = -i \sin it.$$

Also note that the hyperbolic trigonometric functions satisfy the differential equations

$$\frac{d}{dt}\sinh t = \cosh t, \quad \frac{d}{dt}\cosh t = \sinh t,$$

which though similar to the differential equations satisfied by the more commonly used trigonometric functions, is absent a minus sign.

3.4.2 Distinct complex-conjugate roots

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We now consider a characteristic equation (3.11) with $b^2 - 4ac < 0$, so the roots occur as complex-conjugate pairs. With

$$\lambda = -\frac{b}{2a}, \quad \mu = \frac{1}{2a}\sqrt{4ac - b^2},$$

the two roots of the characteristic equation are $\lambda + i\mu$ and $\lambda - i\mu$. We have thus found the following two complex exponential solutions to the differential equation:

$$Z_1(t) = e^{\lambda t}e^{i\mu t}, \quad Z_2(t) = e^{\lambda t}e^{-i\mu t}.$$

Applying the principle of superposition, any linear combination of Z_1 and Z_2 is also a solution to the second-order ode.

Recall that if $z = x + iy$, then $\operatorname{Re} z = (z + \bar{z})/2$ and $\operatorname{Im} z = (z - \bar{z})/2i$. We can therefore form two different linear combinations of $Z_1(t)$ and $Z_2(t)$ that are real, namely $X_1(t) = \operatorname{Re} Z_1(t)$ and $X_2(t) = \operatorname{Im} Z_1(t)$. We have

$$X_1(t) = e^{\lambda t}\cos\mu t, \quad X_2(t) = e^{\lambda t}\sin\mu t.$$

Having found these two real solutions, $X_1(t)$ and $X_2(t)$, we can then apply the principle of superposition a second time to determine the general solution for $x(t)$:

$$x(t) = e^{\lambda t}\left(A\cos\mu t + B\sin\mu t\right). \tag{3.12}$$

It is best to memorize this result. The real part of the roots of the characteristic equation goes into the exponential function; the imaginary part goes into the argument of cosine and sine.

Example 1: Solve $\ddot{x} + x = 0$ with $x(0) = x_0$ and $\dot{x}(0) = u_0$.

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The characteristic equation is

$$r^2 + 1 = 0,$$

with roots

$$r_\pm = \pm i.$$

The general solution of the ode is therefore

$$x(t) = A\cos t + B\sin t.$$

The derivative is

$$\dot{x}(t) = -A\sin t + B\cos t.$$

Applying the initial conditions:

$$x(0) = A = x_0, \quad \dot{x}(0) = B = u_0;$$

so that the final solution is

$$x(t) = x_0\cos t + u_0\sin t.$$

Recall that we wrote the analogous solution to the ode $\ddot{x} - x = 0$ as $x(t) = x_0\cosh t + u_0\sinh t$.

Example 2: Solve $\ddot{x} + \dot{x} + x = 0$ with $x(0) = 1$ and $\dot{x}(0) = 0$.

The characteristic equation is
$$r^2 + r + 1 = 0,$$
with roots
$$r_\pm = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$
The general solution of the ode is therefore
$$x(t) = e^{-\frac{1}{2}t}\left(A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right).$$

The derivative is
$$\dot{x}(t) = -\frac{1}{2}e^{-\frac{1}{2}t}\left(A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2}e^{-\frac{1}{2}t}\left(-A\sin\frac{\sqrt{3}}{2}t + B\cos\frac{\sqrt{3}}{2}t\right).$$

Applying the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$:
$$A = 1,$$
$$\frac{-1}{2}A + \frac{\sqrt{3}}{2}B = 0;$$
or
$$A = 1, \quad B = \frac{\sqrt{3}}{3}.$$
Therefore,
$$x(t) = e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t\right).$$

3.4.3 Repeated roots

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Finally, we consider the characteristic equation,
$$ar^2 + br + c = 0,$$
with $b^2 - 4ac = 0$. The degenerate root is then given by
$$r = -\frac{b}{2a},$$
yielding only a single solution to the ode:
$$x_1(t) = \exp\left(-\frac{bt}{2a}\right). \tag{3.13}$$

To satisfy two initial conditions, a second independent solution must be found with nonzero Wronskian, and apparently this second solution is not of the form of our ansatz $x = \exp(rt)$.

One method to determine this missing second solution is to try the ansatz

$$x(t) = y(t)x_1(t), \tag{3.14}$$

where $y(t)$ is an unknown function that satisfies the differential equation obtained by substituting (3.14) into (3.9). This standard technique is called the reduction of order method and enables one to find a second solution of a homogeneous linear differential equation if one solution is known. If the original differential equation is of order n, the differential equation for $y = y(t)$ reduces to an order one lower, that is, $n - 1$.

Here, however, we will determine this missing second solution through a limiting process. We start with the solution obtained for complex roots of the characteristic equation, and then arrive at the solution obtained for degenerate roots by taking the limit $\mu \to 0$.

Now, the general solution for complex roots was given by (3.12), and to properly limit this solution as $\mu \to 0$ requires first satisfying the specific initial conditions $x(0) = x_0$ and $\dot{x}(0) = u_0$. Solving for A and B, the general solution given by (3.12) becomes the specific solution

$$x(t;\mu) = e^{\lambda t}\left(x_0 \cos \mu t + \frac{u_0 - \lambda x_0}{\mu} \sin \mu t\right).$$

Here, we have written $x = x(t;\mu)$ to show explicitly that x depends on μ.

Taking the limit as $\mu \to 0$, and using $\lim_{\mu \to 0} \mu^{-1} \sin \mu t = t$, we have

$$\lim_{\mu \to 0} x(t;\mu) = e^{\lambda t}\big(x_0 + (u_0 - \lambda x_0)t\big).$$

The second solution is observed to be a constant, $u_0 - \lambda x_0$, times t times the first solution, $e^{\lambda t}$. Our general solution to the ode (3.9) when $b^2 - 4ac = 0$ can therefore be written in the form

$$x(t) = (c_1 + c_2 t)e^{rt},$$

where r is the repeated root of the characteristic equation. The main result to be remembered is that for the case of repeated roots, the second solution is t times the first solution.

Example: Solve $\ddot{x} + 2\dot{x} + x = 0$ with $x(0) = 1$ and $\dot{x}(0) = 0$.

The characteristic equation is

$$\begin{aligned} r^2 + 2r + 1 &= (r+1)^2 \\ &= 0, \end{aligned}$$

which has a repeated root given by $r = -1$. Therefore, the general solution to the ode is

$$x(t) = c_1 e^{-t} + c_2 t e^{-t},$$

with derivative

$$\dot{x}(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}.$$

Applying the initial conditions, we have

$$\begin{aligned} c_1 &= 1, \\ -c_1 + c_2 &= 0, \end{aligned}$$

so that $c_1 = c_2 = 1$. Therefore, the solution is

$$x(t) = (1 + t)e^{-t}.$$

3.5 Inhomogeneous linear second-order ode

We now consider the general inhomogeneous linear second-order ode (3.1):

$$\ddot{x} + p(t)\dot{x} + q(t)x = g(t), \tag{3.15}$$

with initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = u_0$. There is a three-step solution method when the inhomogeneous term $g(t) \neq 0$. (i) Find the general solution of the homogeneous equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0. \tag{3.16}$$

Let us denote the homogeneous solution by

$$x_h(t) = c_1 X_1(t) + c_2 X_2(t),$$

where X_1 and X_2 are linearly independent solutions of (3.16), and c_1 and c_2 are as yet undetermined constants. (ii) Find any *particular* solution x_p of the inhomogeneous equation (3.15). A particular solution is readily found when $p(t)$ and $q(t)$ are constants, and when $g(t)$ is a combination of polynomials, exponentials, sines and cosines. (iii) Write the general solution of (3.15) as the sum of the homogeneous and particular solutions,

$$x(t) = x_h(t) + x_p(t), \tag{3.17}$$

and apply the initial conditions to determine the constants c_1 and c_2. Note that because of the linearity of (3.15),

$$\begin{aligned}\ddot{x} + p\dot{x} + qx &= \frac{d^2}{dt^2}(x_h + x_p) + p\frac{d}{dt}(x_h + x_p) + q(x_h + x_p)\\ &= (\ddot{x}_h + p\dot{x}_h + qx_h) + (\ddot{x}_p + p\dot{x}_p + qx_p)\\ &= 0 + g\\ &= g,\end{aligned}$$

so that (3.17) solves (3.15), and the two free constants in x_h can be used to satisfy the initial conditions.

We will consider here only the constant coefficient case. We now illustrate the solution method by an example.

Example: Solve $\ddot{x} - 3\dot{x} - 4x = 3e^{2t}$ with $x(0) = 1$ and $\dot{x}(0) = 0$.

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First, we solve the homogeneous equation. The characteristic equation is

$$\begin{aligned}r^2 - 3r - 4 &= (r-4)(r+1)\\ &= 0,\end{aligned}$$

so that

$$x_h(t) = c_1 e^{4t} + c_2 e^{-t}.$$

Second, we find a particular solution of the inhomogeneous equation. The form of the particular solution is chosen such that the exponential will cancel out of both sides of the ode. The ansatz we choose is

$$x(t) = Ae^{2t}, \tag{3.18}$$

where A is a yet undetermined coefficient. Upon substituting x into the ode, differentiating using the chain rule, and canceling the exponential, we obtain

$$4A - 6A - 4A = 3,$$

from which we determine $A = -1/2$. Obtaining a solution for A independent of t justifies the ansatz (3.18). Third, we write the general solution to the ode as the sum of the homogeneous and particular solutions, and determine c_1 and c_2 that satisfy the initial conditions. We have

$$x(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t};$$

and taking the derivative,

$$\dot{x}(t) = 4c_1 e^{4t} - c_2 e^{-t} - e^{2t}.$$

Applying the initial conditions,

$$\begin{aligned} c_1 + c_2 - \frac{1}{2} &= 1, \\ 4c_1 - c_2 - 1 &= 0; \end{aligned}$$

or

$$\begin{aligned} c_1 + c_2 &= \frac{3}{2}, \\ 4c_1 - c_2 &= 1. \end{aligned}$$

This system of linear equations can be solved for c_1 by adding the equations to obtain $c_1 = 1/2$, after which $c_2 = 1$ can be determined from the first equation. Therefore, the solution for $x(t)$ that satisfies both the ode and the initial conditions is given by

$$\begin{aligned} x(t) &= \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t} + e^{-t} \\ &= \frac{1}{2}e^{4t}\left(1 - e^{-2t} + 2e^{-5t}\right), \end{aligned}$$

where we have grouped the terms in the solution to better display the asymptotic behavior for large t.

We now find particular solutions for some relatively simple inhomogeneous terms using this method of undetermined coefficients.

Example: Find a particular solution of $\ddot{x} - 3\dot{x} - 4x = 2\sin t$.

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We show two methods for finding a particular solution. The first more direct method tries the ansatz

$$x(t) = A\cos t + B\sin t,$$

where the argument of cosine and sine must agree with the argument of sine in the inhomogeneous term. The cosine term is required because the derivative of sine is cosine. Upon substitution into the differential equation, we obtain

$$(-A\cos t - B\sin t) - 3(-A\sin t + B\cos t) - 4(A\cos t + B\sin t) = 2\sin t,$$

or regrouping terms,

$$-(5A+3B)\cos t+(3A-5B)\sin t=2\sin t.$$

This equation is valid for all t, and in particular for $t=0$ and $\pi/2$, for which the sine and cosine functions vanish. For these two values of t, we find

$$5A+3B=0, \quad 3A-5B=2;$$

and solving for A and B, we obtain

$$A=\frac{3}{17}, \quad B=-\frac{5}{17}.$$

The particular solution is therefore given by

$$x_p=\frac{1}{17}\left(3\cos t-5\sin t\right).$$

The second solution method makes use of the relation $e^{it}=\cos t+i\sin t$ to convert the sine inhomogeneous term to an exponential function. We introduce the complex function $z(t)$ by letting

$$z(t)=x(t)+iy(t),$$

and rewrite the differential equation in complex form. We can rewrite the equation in one of two ways. On the one hand, if we use $\sin t=\mathrm{Re}\{-ie^{it}\}$, then the differential equation is written as

$$\ddot{z}-3\dot{z}-4z=-2ie^{it}; \tag{3.19}$$

and by equating the real and imaginary parts, this equation becomes the two real differential equations

$$\ddot{x}-3\dot{x}-4x=2\sin t, \quad \ddot{y}-3\dot{y}-4y=-2\cos t.$$

The solution we are looking for, then, is $x_p(t)=\mathrm{Re}\{z_p(t)\}$.

On the other hand, if we write $\sin t=\mathrm{Im}\{e^{it}\}$, then the complex differential equation becomes

$$\ddot{z}-3\dot{z}-4z=2e^{it}, \tag{3.20}$$

which becomes the two real differential equations

$$\ddot{x}-3\dot{x}-4x=2\cos t, \quad \ddot{y}-3\dot{y}-4y=2\sin t.$$

Here, the solution we are looking for is $x_p(t)=\mathrm{Im}\{z_p(t)\}$.

We will proceed here by solving (3.20). As we now have an exponential function as the inhomogeneous term, we can make the ansatz

$$z(t)=Ce^{it},$$

where we now expect C to be a complex constant. Upon substitution into the ode (3.20) and using $i^2=-1$:

$$-C-3iC-4C=2;$$

or solving for C:

$$\begin{aligned} C &= \frac{-2}{5+3i} \\ &= \frac{-2(5-3i)}{(5+3i)(5-3i)} \\ &= \frac{-10+6i}{34} \\ &= \frac{-5+3i}{17}. \end{aligned}$$

Therefore,

$$\begin{aligned} x_p &= \mathrm{Im}\{z_p\} \\ &= \mathrm{Im}\left\{\frac{1}{17}(-5+3i)(\cos t + i \sin t)\right\} \\ &= \frac{1}{17}(3\cos t - 5\sin t). \end{aligned}$$

Example: Find a particular solution of $\ddot{x} + \dot{x} - 2x = t^2$.

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The correct ansatz here is the polynomial

$$x(t) = At^2 + Bt + C.$$

Upon substitution into the ode, we have

$$2A + 2At + B - 2At^2 - 2Bt - 2C = t^2,$$

or

$$-2At^2 + 2(A-B)t + (2A + B - 2C)t^0 = t^2.$$

Equating powers of t,

$$-2A = 1, \quad 2(A-B) = 0, \quad 2A + B - 2C = 0;$$

and solving,

$$A = -\frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = -\frac{3}{4}.$$

The particular solution is therefore

$$x_p(t) = -\frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}.$$

3.6 Inhomogeneous linear first-order odes revisited

The linear first-order ode can be solved by use of an integrating factor. Here I show that odes having constant coefficients can be solved by our newly learned solution method.

Example: Solve $\dot{x} + 2x = e^{-t}$ with $x(0) = 3/4$.

Rather than using an integrating factor, we follow the three-step approach: (i) find the general homogeneous solution; (ii) find a particular solution; (iii) add them and satisfy initial conditions. Accordingly, we try the ansatz $x_h(t) = e^{rt}$ for the homogeneous ode $\dot{x} + 2x = 0$ and find

$$r + 2 = 0, \quad \text{or} \quad r = -2.$$

To find a particular solution, we try the ansatz $x_p(t) = Ae^{-t}$, and upon substitution

$$-A + 2A = 1, \quad \text{or} \quad A = 1.$$

Therefore, the general solution to the ode is

$$x(t) = ce^{-2t} + e^{-t}.$$

The single initial condition determines the unknown constant c:

$$x(0) = \frac{3}{4} = c + 1,$$

so that $c = -1/4$. Hence,

$$\begin{aligned} x(t) &= e^{-t} - \frac{1}{4}e^{-2t} \\ &= e^{-t}\left(1 - \frac{1}{4}e^{-t}\right). \end{aligned}$$

3.7 Resonance

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Resonance occurs when the frequency of the inhomogeneous term matches the frequency of the homogeneous solution. To illustrate resonance in its simplest embodiment, we consider the second-order linear inhomogeneous ode

$$\ddot{x} + \omega_0^2 x = f\cos\omega t, \quad x(0) = x_0,\ \dot{x}(0) = u_0. \tag{3.21}$$

Our main goal is to determine what happens to the solution in the limit $\omega \to \omega_0$.

The homogeneous equation has characteristic equation

$$r^2 + \omega_0^2 = 0,$$

so that $r_\pm = \pm i\omega_0$. Therefore,

$$x_h(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t. \tag{3.22}$$

To find a particular solution, we note the absence of a first-derivative term, and simply try

$$x(t) = A\cos\omega t.$$

Upon substitution into the ode, we obtain

$$-\omega^2 A + \omega_0^2 A = f,$$

or

$$A = \frac{f}{\omega_0^2 - \omega^2}.$$

Therefore,

$$x_p(t) = \frac{f}{\omega_0^2 - \omega^2} \cos \omega t.$$

Our general solution is thus

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{f}{\omega_0^2 - \omega^2} \cos \omega t,$$

with derivative

$$\dot{x}(t) = \omega_0 (c_2 \cos \omega_0 t - c_1 \sin \omega_0 t) - \frac{f \omega}{\omega_0^2 - \omega^2} \sin \omega t.$$

Initial conditions are satisfied when

$$\begin{aligned} x_0 &= c_1 + \frac{f}{\omega_0^2 - \omega^2}, \\ u_0 &= c_2 \omega_0, \end{aligned}$$

so that

$$c_1 = x_0 - \frac{f}{\omega_0^2 - \omega^2}, \quad c_2 = \frac{u_0}{\omega_0}.$$

Therefore, the solution to the ode that satisfies the initial conditions is

$$\begin{aligned} x(t) &= \left(x_0 - \frac{f}{\omega_0^2 - \omega^2} \right) \cos \omega_0 t + \frac{u_0}{\omega_0} \sin \omega_0 t + \frac{f}{\omega_0^2 - \omega^2} \cos \omega t \\ &= x_0 \cos \omega_0 t + \frac{u_0}{\omega_0} \sin \omega_0 t + \frac{f(\cos \omega t - \cos \omega_0 t)}{\omega_0^2 - \omega^2}, \end{aligned}$$

where we have grouped together terms proportional to the forcing amplitude f.

Resonance occurs in the limit $\omega \to \omega_0$; that is, the frequency of the inhomogeneous term (the external force) matches the frequency of the homogeneous solution (the free oscillation). By L'Hospital's rule, the limit of the term proportional to f is found by differentiating with respect to ω:

$$\begin{aligned} \lim_{\omega \to \omega_0} \frac{f(\cos \omega t - \cos \omega_0 t)}{\omega_0^2 - \omega^2} &= \lim_{\omega \to \omega_0} \frac{-ft \sin \omega t}{-2\omega} \\ &= \frac{ft \sin \omega_0 t}{2\omega_0}. \end{aligned} \tag{3.23}$$

At resonance, the term proportional to the amplitude f of the inhomogeneous term increases linearly with t, resulting in larger-and-larger amplitudes of oscillation for $x(t)$. In general, if the inhomogeneous term in the differential equation is a solution of the corresponding homogeneous differential equation, then the correct ansatz for the particular solution is a constant times the inhomogeneous term times t.

To illustrate this same example further, we return to the original ode, now assumed to be exactly at resonance,

$$\ddot{x} + \omega_0^2 x = f \cos \omega_0 t,$$

and find a particular solution directly. The particular solution is the real part of the particular solution of

$$\ddot{z} + \omega_0^2 z = f e^{i\omega_0 t}.$$

If we try $z_p = Ce^{i\omega_0 t}$, we obtain $0 = f$, showing that the particular solution is not of this form. Because the inhomogeneous term is a solution of the homogeneous equation, we should take as our ansatz

$$z_p = Ate^{i\omega_0 t}.$$

We have

$$\dot{z}_p = Ae^{i\omega_0 t}\left(1 + i\omega_0 t\right), \quad \ddot{z}_p = Ae^{i\omega_0 t}\left(2i\omega_0 - \omega_0^2 t\right);$$

and upon substitution into the ode

$$\begin{aligned}\ddot{z}_p + \omega_0^2 z_p &= Ae^{i\omega_0 t}\left(2i\omega_0 - \omega_0^2 t\right) + \omega_0^2 Ate^{i\omega_0 t} \\ &= 2i\omega_0 Ae^{i\omega_0 t} \\ &= fe^{i\omega_0 t}.\end{aligned}$$

Therefore,

$$A = \frac{f}{2i\omega_0},$$

and

$$\begin{aligned}x_p &= \text{Re}\{\frac{ft}{2i\omega_0}e^{i\omega_0 t}\} \\ &= \frac{ft\sin\omega_0 t}{2\omega_0},\end{aligned}$$

the same result as (3.23).

Example: Find a particular solution of $\ddot{x} - 3\dot{x} - 4x = 5e^{-t}$.

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If we naively try the ansatz

$$x = Ae^{-t},$$

and substitute this into the inhomogeneous differential equation, we obtain

$$A + 3A - 4A = 5,$$

or $0 = 5$, which is clearly nonsense. Our ansatz therefore fails to find a solution. The cause of this failure is that the corresponding homogeneous equation has solution

$$x_h = c_1 e^{4t} + c_2 e^{-t},$$

so that the inhomogeneous term $5e^{-t}$ is one of the solutions of the homogeneous equation. To find a particular solution, we should therefore take as our ansatz

$$x = Ate^{-t},$$

with first- and second-derivatives given by

$$\dot{x} = Ae^{-t}(1-t), \quad \ddot{x} = Ae^{-t}(-2+t).$$

Substitution into the differential equation yields

$$Ae^{-t}(-2+t) - 3Ae^{-t}(1-t) - 4Ate^{-t} = 5e^{-t}.$$

The terms containing t cancel out of this equation, resulting in $-5A = 5$, or $A = -1$. Therefore, the particular solution is

$$x_p = -te^{-t}.$$

3.8 Damped resonance

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A more realistic study of resonance assumes an additional damping term. The forced, damped harmonic oscillator equation may be written as

$$m\ddot{x} + \gamma\dot{x} + kx = F\cos\omega t, \tag{3.24}$$

where $m > 0$ is the oscillator's mass, $\gamma > 0$ is the damping coefficient, $k > 0$ is the spring constant, and F is the amplitude of the external force. The homogeneous equation has characteristic equation

$$mr^2 + \gamma r + k = 0,$$

so that

$$r_\pm = -\frac{\gamma}{2m} \pm \frac{1}{2m}\sqrt{\gamma^2 - 4mk}.$$

When $\gamma^2 - 4mk < 0$, the motion of the unforced oscillator is said to be underdamped; when $\gamma^2 - 4mk > 0$, overdamped; and when $\gamma^2 - 4mk = 0$, critically damped. For all three types of damping, the roots of the characteristic equation satisfy $\text{Re}(r_\pm) < 0$. Therefore, both linearly independent homogeneous solutions decay exponentially to zero, and the long-time asymptotic solution of (3.24) reduces to the non-decaying particular solution. Since the initial conditions are satisfied by the free constants multiplying the decaying homogeneous solutions, the long-time asymptotic solution is independent of the initial conditions.

If we are only interested in the long-time asymptotic solution of (3.24), we can proceed directly to the determination of a particular solution. As before, we consider the complex ode

$$m\ddot{z} + \gamma\dot{z} + kz = Fe^{i\omega t},$$

with $x_p = \text{Re}(z_p)$. With the ansatz $z_p = Ae^{i\omega t}$, we have

$$-m\omega^2 A + i\gamma\omega A + kA = F,$$

or

$$A = \frac{F}{(k - m\omega^2) + i\gamma\omega}.$$

To simplify, we define $\omega_0 = \sqrt{k/m}$, which corresponds to the natural frequency of the undamped oscillator, and define $\Gamma = \gamma\omega/m$ and $f = F/m$. Therefore,

$$\begin{aligned} A &= \frac{f}{(\omega_0^2 - \omega^2) + i\Gamma} \\ &= \left(\frac{f}{(\omega_0^2 - \omega^2)^2 + \Gamma^2}\right)\left((\omega_0^2 - \omega^2) - i\Gamma\right). \end{aligned} \tag{3.25}$$

To determine x_p, we utilize the polar form of a complex number. The complex number $z = x + iy$ can be written in polar form as $z = re^{i\phi}$, where $x = r\cos\phi$, $y = r\sin\phi$, and $r = \sqrt{x^2 + y^2}$, $\tan\phi = y/x$. We therefore write

$$(\omega_0^2 - \omega^2) - i\Gamma = re^{i\phi},$$

with

$$r = \sqrt{(\omega_0^2 - \omega^2)^2 + \Gamma^2}, \quad \tan\phi = \frac{\Gamma}{(\omega^2 - \omega_0^2)}.$$

Using the polar form, A in (3.25) becomes

$$A = \left(\frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + \Gamma^2}} \right) e^{i\phi},$$

and $x_p = \mathrm{Re}(Ae^{i\omega t})$ becomes

$$\begin{aligned} x_p &= \left(\frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + \Gamma^2}} \right) \mathrm{Re}\left(e^{i(\omega t + \phi)}\right) \\ &= \left(\frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + \Gamma^2}} \right) \cos(\omega t + \phi). \end{aligned} \tag{3.26}$$

The particular solution given by (3.26), with $\omega_0^2 = k/m$, $\Gamma = \gamma\omega/m$, $f = F/m$, and $\tan\phi = \gamma\omega/m(\omega^2 - \omega_0^2)$, is the long-time asymptotic solution of the forced, damped, harmonic oscillator equation (3.24).

We conclude with a couple of observations about (3.26). First, if the forcing frequency ω is equal to the natural frequency ω_0 of the undamped oscillator, then $A = -iF/\gamma\omega_0$, and $x_p = (F/\gamma\omega_0)\sin\omega_0 t$. The oscillator position is observed to be $\pi/2$ out of phase with the external force, or in other words, the velocity of the oscillator, not the position, is in phase with the force. Second, the value of the forcing frequency ω_m that maximizes the amplitude of oscillation is the value of ω that minimizes the denominator of (3.26). To determine ω_m we thus minimize the function $g(\omega^2)$ with respect to ω^2, where

$$g(\omega^2) = (\omega_0^2 - \omega^2)^2 + \frac{\gamma^2\omega^2}{m^2}.$$

Taking the derivative of g with respect to ω^2 and setting this to zero to determine ω_m yields

$$2(\omega_m^2 - \omega_0^2) + \frac{\gamma^2}{m^2} = 0,$$

or

$$\omega_m^2 = \omega_0^2 - \frac{\gamma^2}{2m^2}.$$

We can interpret this result by saying that damping lowers the “resonance” frequency of the undamped oscillator.

Chapter 4

The Laplace transform

Reference: Boyce and DiPrima, Chapter 6

The Laplace transform is most useful for solving linear, constant-coefficient ode's when the inhomogeneous term or its derivative is discontinuous. Although ode's with discontinuous inhomogeneous terms can also be solved by adopting already learned methods, we will see that the Laplace transform technique provides a simpler, more elegant solution.

4.1 Definitions and properties of the forward and inverse Laplace transforms

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The main idea is to Laplace transform the constant-coefficient differential equation for $x(t)$ into a simpler algebraic equation for the Laplace-transformed function $X(s)$, solve this algebraic equation, and then transform $X(s)$ back into $x(t)$. The correct definition of the Laplace transform and the properties that this transform satisfies makes this solution method possible.

An exponential ansatz is used in solving homogeneous constant-coefficient odes, and the exponential function correspondingly plays a key role in defining the Laplace transform. The Laplace transform of $f(t)$, denoted by $F(s) = \mathcal{L}\{f(t)\}$, is defined by the integral transform

$$F(s) = \int_0^\infty e^{-st} f(t) dt. \tag{4.1}$$

The improper integral given by (4.1) diverges if $f(t)$ grows faster than e^{st} for large t. Accordingly, some restriction on the range of s may be required to guarantee convergence of (4.1), and we will assume without further elaboration that these restrictions are always satisfied.

The Laplace transform is a linear transformation. We have

$$\begin{aligned}\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_0^\infty e^{-st}\big(c_1 f_1(t) + c_2 f_2(t)\big)dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t)dt + c_2 \int_0^\infty e^{-st} f_2(t)dt \\ &= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}.\end{aligned}$$

There is also a one-to-one correspondence between functions and their Laplace transforms. A table of Laplace transforms can therefore be constructed and used to find both Laplace and inverse Laplace transforms of commonly occurring functions. Such a table is shown in Table 4.1 (and this table will be distributed with the exams). In Table 4.1, n is a positive integer. Also, the cryptic entries for $u_c(t)$ and $\delta(t-c)$ will be explained later in §4.3.

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. $e^{at}f(t)$	$F(s-a)$
2. 1	$\frac{1}{s}$
3. e^{at}	$\frac{1}{s-a}$
4. t^n	$\frac{n!}{s^{n+1}}$
5. $t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
6. $\sin bt$	$\frac{b}{s^2+b^2}$
7. $\cos bt$	$\frac{s}{s^2+b^2}$
8. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
9. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
10. $t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
11. $t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
12. $u_c(t)$	$\frac{e^{-cs}}{s}$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $\delta(t-c)$	e^{-cs}
15. $\dot{x}(t)$	$sX(s)-x(0)$
16. $\ddot{x}(t)$	$s^2X(s)-sx(0)-\dot{x}(0)$

Table 4.1: Table of Laplace Transforms

The rows of Table 4.1 can be determined by a combination of direct integration and some tricks. We first compute directly the Laplace transform of $e^{at}f(t)$ (line 1):

$$\begin{aligned} \mathcal{L}\{e^{at}f(t)\} &= \int_0^\infty e^{-st}e^{at}f(t)dt \\ &= \int_0^\infty e^{-(s-a)t}f(t)dt \\ &= F(s-a). \end{aligned}$$

We also compute directly the Laplace transform of 1 (line 2):

$$\begin{aligned} \mathcal{L}\{1\} &= \int_0^\infty e^{-st}dt \\ &= -\frac{1}{s}e^{-st}\Big]_0^\infty \\ &= \frac{1}{s}. \end{aligned}$$

Now, the Laplace transform of e^{at} (line 3) may be found using these two results:

$$\begin{aligned} \mathcal{L}\{e^{at}\} &= \mathcal{L}\{e^{at}\cdot 1\} \\ &= \frac{1}{s-a}. \end{aligned} \tag{4.2}$$

The transform of t^n (line 4) can be found by successive integration by parts. A more interesting method uses Taylor series expansions for e^{at} and $1/(s-a)$. We have

$$\begin{aligned} \mathcal{L}\{e^{at}\} &= \mathcal{L}\left\{\sum_{n=0}^\infty \frac{(at)^n}{n!}\right\} \\ &= \sum_{n=0}^\infty \frac{a^n}{n!}\mathcal{L}\{t^n\}. \end{aligned} \tag{4.3}$$

Also, with $s > a$,

$$\begin{aligned} \frac{1}{s-a} &= \frac{1}{s(1-\frac{a}{s})} \\ &= \frac{1}{s}\sum_{n=0}^\infty \left(\frac{a}{s}\right)^n \\ &= \sum_{n=0}^\infty \frac{a^n}{s^{n+1}}. \end{aligned} \tag{4.4}$$

Using (4.2), and equating the coefficients of powers of a in (4.3) and (4.4), results in line 4:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

The Laplace transform of $t^n e^{at}$ (line 5) can be found from line 1 and line 4:

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}.$$

The Laplace transform of $\sin bt$ (line 6) may be found from the Laplace transform of e^{at} (line 3) using $a = ib$:

$$\begin{aligned}\mathcal{L}\{\sin bt\} &= \operatorname{Im}\{\mathcal{L}\{e^{ibt}\}\} \\ &= \operatorname{Im}\left\{\frac{1}{s-ib}\right\} \\ &= \operatorname{Im}\left\{\frac{s+ib}{s^2+b^2}\right\} \\ &= \frac{b}{s^2+b^2}.\end{aligned}$$

Similarly, the Laplace transform of $\cos bt$ (line 7) is

$$\begin{aligned}\mathcal{L}\{\cos bt\} &= \operatorname{Re}\{\mathcal{L}\{e^{ibt}\}\} \\ &= \frac{s}{s^2+b^2}.\end{aligned}$$

The transform of $e^{at}\sin bt$ (line 8) can be found from the transform of $\sin bt$ (line 6) and line 1:

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2};$$

and similarly for the transform of $e^{at}\cos bt$:

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}.$$

The Laplace transform of $t\sin bt$ (line 10) can be found from the Laplace transform of te^{at} (line 5 with $n = 1$) using $a = ib$:

$$\begin{aligned}\mathcal{L}\{t\sin bt\} &= \operatorname{Im}\{\mathcal{L}\{te^{ibt}\}\} \\ &= \operatorname{Im}\left\{\frac{1}{(s-ib)^2}\right\} \\ &= \operatorname{Im}\left\{\frac{(s+ib)^2}{(s^2+b^2)^2}\right\} \\ &= \frac{2bs}{(s^2+b^2)^2}.\end{aligned}$$

Similarly, the Laplace transform of $t\cos bt$ (line 11) is

$$\begin{aligned}\mathcal{L}\{t\cos bt\} &= \operatorname{Re}\{\mathcal{L}\{te^{ibt}\}\} \\ &= \operatorname{Re}\left\{\frac{(s+ib)^2}{(s^2+b^2)^2}\right\} \\ &= \frac{s^2-b^2}{(s^2+b^2)^2}.\end{aligned}$$

We now transform the inhomogeneous constant-coefficient, second-order, linear inhomogeneous ode for $x = x(t)$,

$$a\ddot{x} + b\dot{x} + cx = g(t),$$

making use of the linearity of the Laplace transform:

$$a\mathcal{L}\{\ddot{x}\} + b\mathcal{L}\{\dot{x}\} + c\mathcal{L}\{x\} = \mathcal{L}\{g\}.$$

To determine the Laplace transform of $\dot{x}(t)$ (line 15) in terms of the Laplace transform of $x(t)$ and the initial conditions $x(0)$ and $\dot{x}(0)$, we define $X(s) = \mathcal{L}\{x(t)\}$, and integrate

$$\int_0^\infty e^{-st}\dot{x}dt$$

by parts. We let

$$\begin{aligned} u &= e^{-st} & dv &= \dot{x}dt \\ du &= -se^{-st}dt & v &= x. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^\infty e^{-st}\dot{x}dt &= xe^{-st}\Big]_0^\infty + s\int_0^\infty e^{-st}xdt \\ &= sX(s) - x(0), \end{aligned}$$

where assumed convergence of the Laplace transform requires

$$\lim_{t\to\infty} x(t)e^{-st} = 0.$$

Similarly, the Laplace transform of $\ddot{x}(t)$ (line 16) is determined by integrating

$$\int_0^\infty e^{-st}\ddot{x}dt$$

by parts and using the just derived result for the first derivative. We let

$$\begin{aligned} u &= e^{-st} & dv &= \ddot{x}dt \\ du &= -se^{-st}dt & v &= \dot{x}, \end{aligned}$$

so that

$$\begin{aligned} \int_0^\infty e^{-st}\ddot{x}dt &= \dot{x}e^{-st}\Big]_0^\infty + s\int_0^\infty e^{-st}\dot{x}dt \\ &= -\dot{x}(0) + s\big(sX(s) - x(0)\big) \\ &= s^2X(s) - sx(0) - \dot{x}(0), \end{aligned}$$

where similarly we assume $\lim_{t\to\infty}\dot{x}(t)e^{-st} = 0$.

4.2 Solution of initial value problems

We begin with a simple homogeneous ode and show that the Laplace transform method yields an identical result to our previously learned method. We then apply the Laplace transform method to solve an inhomogeneous equation.

Example: Solve $\ddot{x} - \dot{x} - 2x = 0$ with $x(0) = 1$ and $\dot{x}(0) = 0$ by two different methods.

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The characteristic equation of the ode is determined from the ansatz $x = e^{rt}$ and is

$$r^2 - r - 2 = (r-2)(r+1) = 0.$$

The general solution of the ode is therefore

$$x(t) = c_1 e^{2t} + c_2 e^{-t}.$$

To satisfy the initial conditions, we must have $1 = c_1 + c_2$ and $0 = 2c_1 - c_2$, requiring $c_1 = \frac{1}{3}$ and $c_2 = \frac{2}{3}$. Therefore, the solution to the ode that satisfies the initial conditions is given by

$$x(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}. \tag{4.5}$$

We now solve this example using the Laplace transform. Taking the Laplace transform of both sides of the ode, using the linearity of the transform, and applying our result for the transform of the first and second derivatives, we find

$$[s^2 X(s) - sx(0) - \dot{x}(0)] - [sX(s) - x(0)] - [2X(s)] = 0,$$

or

$$X(s) = \frac{(s-1)x(0) + \dot{x}(0)}{s^2 - s - 2}.$$

Note that the denominator of the right-hand-side is just the quadratic from the characteristic equation of the homogeneous ode, and that this factor arises from the derivatives of the exponential term in the Laplace transform integral.

Applying the initial conditions, we find

$$X(s) = \frac{s-1}{(s-2)(s+1)}. \tag{4.6}$$

We have thus determined the Laplace transformed solution $X(s) = \mathcal{L}\{x(t)\}$. We now need to compute the inverse Laplace transform $x(t) = \mathcal{L}^{-1}\{X(s)\}$.

However, direct inversion of (4.6) by searching Table 4.1 is not possible, but a partial fraction expansion may be useful. In particular, we write

$$\frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}. \tag{4.7}$$

The cover-up method can be used to solve for a and b. We multiply both sides of (4.7) by $s-2$ and put $s = 2$ to isolate a:

$$\begin{aligned} a &= \left.\frac{s-1}{s+1}\right]_{s=2} \\ &= \frac{1}{3}. \end{aligned}$$

Similarly, we multiply both sides of (4.7) by $s+1$ and put $s = -1$ to isolate b:

$$\begin{aligned} b &= \left.\frac{s-1}{s-2}\right]_{s=-1} \\ &= \frac{2}{3}. \end{aligned}$$

Therefore,

$$X(s) = \frac{1}{3} \cdot \frac{1}{s-2} + \frac{2}{3} \cdot \frac{1}{s+1},$$

and line 3 of Table 4.1 gives us the inverse transforms of each term separately to yield

$$x(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t},$$

identical to (4.5).

Example: Solve $\ddot{x} + x = \sin 2t$ with $x(0) = 2$ and $\dot{x}(0) = 1$ by Laplace transform methods.

Taking the Laplace transform of both sides of the ode, we find

$$\begin{aligned} s^2X(s) - sx(0) - \dot{x}(0) + X(s) &= \mathcal{L}\{\sin 2t\} \\ &= \frac{2}{s^2+4}, \end{aligned}$$

where the Laplace transform of $\sin 2t$ made use of line 6 of Table 4.1. Substituting for $x(0)$ and $\dot{x}(0)$ and solving for $X(s)$, we obtain

$$X(s) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}.$$

To determine the inverse Laplace transform from Table 4.1, we perform a partial fraction expansion of the second term:

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}. \tag{4.8}$$

By inspection, we can observe that $a = c = 0$ and that $d = -b$. A quick calculation shows that $3b = 2$, or $b = 2/3$. Therefore,

$$\begin{aligned} X(s) &= \frac{2s+1}{s^2+1} + \frac{2/3}{s^2+1} - \frac{2/3}{(s^2+4)} \\ &= \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{(s^2+4)}. \end{aligned}$$

From lines 6 and 7 of Table 4.1, we obtain the solution by taking inverse Laplace transforms of the three terms separately, where $b = 1$ in the first two terms, and $b = 2$ in the third term:

$$x(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.$$

4.3 Heaviside and Dirac delta functions

The Laplace transform technique becomes truly useful when solving odes with discontinuous or impulsive inhomogeneous terms, these terms commonly modeled using Heaviside or Dirac delta functions. We will discuss these functions in turn, as well as their Laplace transforms.

4.3.1 Heaviside function

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The Heaviside or unit step function, denoted here by $u_c(t)$, is zero for $t < c$ and is one for $t \geq c$; that is,

$$u_c(t) = \begin{cases} 0, & t < c; \\ 1, & t \geq c. \end{cases} \tag{4.9}$$

The precise value of $u_c(t)$ at the single point $t = c$ shouldn't matter.

The Heaviside function can be viewed as the step-up function. The step-down function—one for $t < c$ and zero for $t \geq c$—is defined as

$$1 - u_c(t) = \begin{cases} 1, & t < c; \\ 0, & t \geq c. \end{cases} \tag{4.10}$$

The step-up, step-down function—zero for $t < a$, one for $a \leq t < b$, and zero for $t \geq b$—is defined as

$$u_a(t) - u_b(t) = \begin{cases} 0, & t < a; \\ 1, & a \leq t < b; \\ 0, & t \geq b. \end{cases} \tag{4.11}$$

The Laplace transform of the Heaviside function is determined by integration:

$$\begin{aligned} \mathcal{L}\{u_c(t)\} &= \int_0^\infty e^{-st} u_c(t) dt \\ &= \int_c^\infty e^{-st} dt \\ &= \frac{e^{-cs}}{s}, \end{aligned}$$

and is given in line 12 of Table 4.1.

The Heaviside function can be used to represent a translation of a function $f(t)$ a distance c in the positive t direction. We have

$$u_c(t) f(t-c) = \begin{cases} 0, & t < c; \\ \text{f(t-c)}, & t \geq c. \end{cases}$$

The Laplace transform is

$$\begin{aligned} \mathcal{L}\{u_c(t) f(t-c)\} &= \int_0^\infty e^{-st} u_c(t) f(t-c) dt \\ &= \int_c^\infty e^{-st} f(t-c) dt \\ &= \int_0^\infty e^{-s(t'+c)} f(t') dt' \\ &= e^{-cs} \int_0^\infty e^{-st'} f(t') dt' \\ &= e^{-cs} F(s), \end{aligned}$$

where we have changed variables to $t' = t - c$. The translation of $f(t)$ a distance c in the positive t direction corresponds to the multiplication of $F(s)$ by the exponential e^{-cs}. This result is shown in line 13 of Table 4.1.

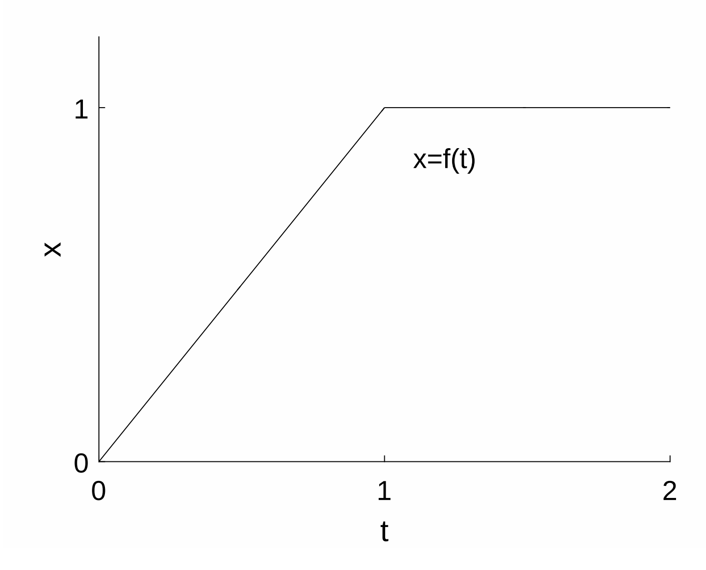


Figure 4.1: *A linearly increasing function which turns into a constant function.*

Piecewise-defined inhomogeneous terms can be modeled using Heaviside functions. For example, consider the general case of a piecewise function defined on two intervals:

$$f(t) = \begin{cases} f_1(t), & \text{if } t < t_*; \\ f_2(t), & \text{if } t \geq t_*. \end{cases}$$

Using the Heaviside function u_{t_*}, the function $f(t)$ can be written in a single line as

$$f(t) = f_1(t) + \big(f_2(t) - f_1(t)\big)u_{t_*}(t).$$

This example can be generalized to piecewise functions defined on multiple intervals.

As a concrete example, suppose the inhomogeneous term is represented by a linearly increasing function, which then turns into a constant function, as sketched in Fig. 4.1. Explicitly,

$$f(t) = \begin{cases} t, & \text{if } t < 1; \\ 1, & \text{if } t \geq 1. \end{cases}$$

We can rewrite $f(t)$ using the Heaviside function $u_1(t)$:

$$f(t) = t - u_1(t)\,(t-1)\,;$$

and we can take the Laplace transform of the function $f(t)$ by writing

$$F(s) = \mathcal{L}\{t\} - \mathcal{L}\{u_1(t)\,(t-1)\}.$$

The Laplace transform of the first term is found from line 4 of the table, and the Laplace transform of the second term is found from a combination of line 13 and line 4. We have

$$F(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2}.$$

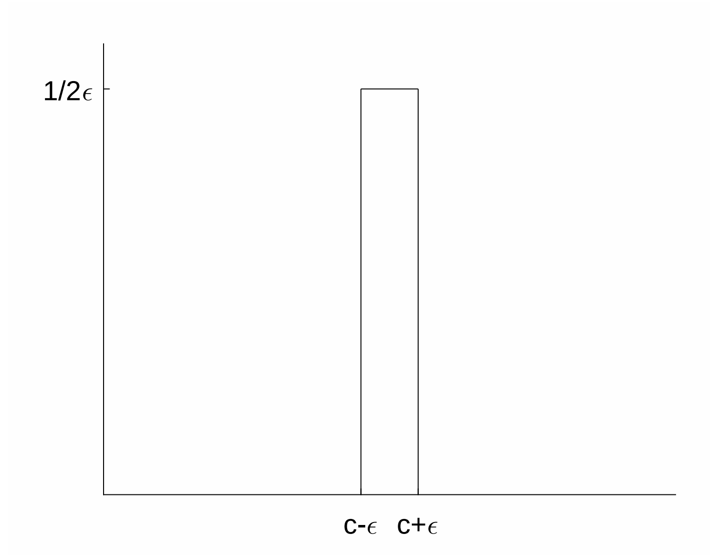


Figure 4.2: *The Dirac delta function constructed using the step-up, step-down function.*

4.3.2 Dirac delta function

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The Dirac delta function, denoted as $\delta(t)$, is defined by requiring that for any function $f(t)$,

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0).$$

The usual view of the shifted Dirac delta function $\delta(t-c)$ is that it is zero everywhere except at $t = c$, where it is infinite, and the integral over the Dirac delta function is one. The Dirac delta function is technically not a function, but is what mathematicians call a distribution. Nevertheless, in most cases of practical interest, it can be treated like a function, where physical results are obtained following a final integration.

There are many ways to represent the Dirac delta function as a limit of a well-defined function. For our purposes, the most useful representation makes use of the step-up, step-down function of (4.11) (see Fig. 4.2):

$$\delta(t-c) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon}(u_{c-\epsilon}(t) - u_{c+\epsilon}(t)).$$

Before taking the limit, the well-defined step-up, step-down function is zero except over a small interval of width 2ϵ centered at $t = c$, over which it takes the large value $1/2\epsilon$. The integral of this function is one, independent of the value of ϵ.

The Laplace transform of the Dirac delta function is easily found by integration using the definition of the delta function:

$$\begin{aligned}\mathcal{L}\{\delta(t-c)\} &= \int_0^{\infty} e^{-st}\delta(t-c)dt \\ &= e^{-cs}.\end{aligned}$$

This result is shown in line 14 of Table 4.1.

4.4 Discontinuous or impulsive inhomogeneous terms

We now solve some more challenging ode's with discontinuous or impulsive inhomogeneous terms.

Example: Solve $2\ddot{x} + \dot{x} + 2x = u_5(t) - u_{20}(t)$**, with** $x(0) = \dot{x}(0) = 0$

The inhomogeneous term here is a step-up, step-down function that is unity over the interval $(5, 20)$ and zero elsewhere. Taking the Laplace transform of the ode using Table 4.1,

$$2\left(s^2X(s) - sx(0) - \dot{x}(0)\right) + sX(s) - x(0) + 2X(s) = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}.$$

Using the initial values and solving for $X(s)$, we find

$$X(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}.$$

To determine the solution for $x(t)$ we now need to find the inverse Laplace transform. The exponential functions can be dealt with using line 13 of Table 4.1. We write

$$X(s) = (e^{-5s} - e^{-20s})H(s),$$

where

$$H(s) = \frac{1}{s(2s^2 + s + 2)}.$$

Then using line 13, we have

$$x(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20), \tag{4.12}$$

where $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

To determine $h(t)$ we need the partial fraction expansion of $H(s)$. Since the discriminant of $2s^2 + s + 2$ is negative, we have

$$\frac{1}{s(2s^2 + s + 2)} = \frac{a}{s} + \frac{bs + c}{2s^2 + s + 2},$$

yielding the equation

$$1 = a(2s^2 + s + 2) + (bs + c)s;$$

or after equating powers of s,

$$2a + b = 0, \quad a + c = 0, \quad 2a = 1,$$

yielding $a = \frac{1}{2}$, $b = -1$, and $c = -\frac{1}{2}$. Therefore,

$$\begin{aligned} H(s) &= \frac{1/2}{s} - \frac{s + \frac{1}{2}}{2s^2 + s + 2} \\ &= \frac{1}{2}\left(\frac{1}{s} - \frac{s + \frac{1}{2}}{s^2 + \frac{1}{2}s + 1}\right). \end{aligned}$$

Inspecting Table 4.1, the first term can be transformed using line 2, and the second term can be transformed using lines 8 and 9, provided we complete the square of

the denominator and then massage the numerator. That is, first we complete the square:

$$s^2 + \frac{1}{2}s + 1 = \left(s + \frac{1}{4}\right)^2 + \frac{15}{16};$$

and next we write

$$\frac{s + \frac{1}{2}}{s^2 + \frac{1}{2}s + 1} = \frac{\left(s + \frac{1}{4}\right) + \frac{1}{\sqrt{15}}\sqrt{\frac{15}{16}}}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}.$$

Therefore, the function $H(s)$ can be written as

$$H(s) = \frac{1}{2}\left(\frac{1}{s} - \frac{\left(s + \frac{1}{4}\right)}{(s + \frac{1}{4})^2 + \frac{15}{16}} - \left(\frac{1}{\sqrt{15}}\right)\frac{\sqrt{\frac{15}{16}}}{(s + \frac{1}{4})^2 + \frac{15}{16}}\right).$$

The first term is transformed using line 2, the second term using line 9 and the third term using line 8. We finally obtain

$$h(t) = \frac{1}{2}\left(1 - e^{-t/4}\left(\cos\left(\sqrt{15}t/4\right) + \frac{1}{\sqrt{15}}\sin\left(\sqrt{15}t/4\right)\right)\right), \tag{4.13}$$

which when combined with (4.12) yields the rather complicated solution for $x(t)$.

We briefly comment that it is also possible to solve this example without using the Laplace transform. The key idea is that both x and $\dot{x}$ are continuous functions of t. Clearly from the form of the inhomogeneous term and the initial conditions, $x(t) = 0$ for $0 \leq t \leq 5$. We then solve the ode between $5 \leq t \leq 20$ with the inhomogeneous term equal to unity and initial conditions $x(5) = \dot{x}(5) = 0$. This requires first finding the general homogeneous solution, next finding a particular solution, and then adding the homogeneous and particular solutions and solving for the two unknown constants. To simplify the algebra, note that the best ansatz to use to find the homogeneous solution is $x(t) = e^{r(t-5)}$, and not $x(t) = e^{rt}$. Finally, we solve the homogeneous ode for $t \geq 20$ using as boundary conditions the previously determined values $x(20)$ and $\dot{x}(20)$, where we have made use of the continuity of x and $\dot{x}$. Here, the best ansatz to use is $x(t) = e^{r(t-20)}$. The student may benefit by trying this as an exercise and attempting to obtain a final solution that agrees with the form given by (4.12) and (4.13).

Example: Solve $2\ddot{x} + \dot{x} + 2x = \delta(t-5)$ with $x(0) = \dot{x}(0) = 0$

Here the inhomogeneous term is an *impulse* at time $t = 5$. Taking the Laplace transform of the ode using Table 4.1, and applying the initial conditions,

$$(2s^2 + s + 2)X(s) = e^{-5s},$$

so that

$$\begin{aligned} X(s) &= \frac{1}{2}e^{-5s}\left(\frac{1}{s^2 + \frac{1}{2}s + 1}\right) \\ &= \frac{1}{2}e^{-5s}\left(\frac{1}{(s + \frac{1}{4})^2 + \frac{15}{16}}\right) \\ &= \frac{1}{2}\sqrt{\frac{16}{15}}e^{-5s}\left(\frac{\sqrt{\frac{15}{16}}}{(s + \frac{1}{4})^2 + \frac{15}{16}}\right). \end{aligned}$$

The inverse Laplace transform may now be computed using lines 8 and 13 of Table 4.1:

$$x(t) = \frac{2}{\sqrt{15}} u_5(t) e^{-(t-5)/4} \sin\left(\sqrt{15}(t-5)/4\right). \tag{4.14}$$

It is interesting to solve this example without using a Laplace transform. Clearly, $x(t) = 0$ up to the time of impulse at $t = 5$. Furthermore, after the impulse the ode is homogeneous and can be solved with standard methods. The only difficulty is determining the initial conditions of the homogeneous ode at $t = 5^+$.

When the inhomogeneous term is proportional to a delta-function, the solution $x(t)$ is continuous across the delta-function singularity, but the derivative of the solution $\dot{x}(t)$ is discontinuous. If we integrate the second-order ode across the singularity at $t = 5$ and consider $\epsilon \to 0$, only the second derivative term of the left-hand-side survives, and

$$\begin{aligned} 2\int_{5-\epsilon}^{5+\epsilon} \ddot{x} dt &= \int_{5-\epsilon}^{5+\epsilon} \delta(t-5) dt \\ &= 1. \end{aligned}$$

And as $\epsilon \to 0$, we have $\dot{x}(5^+) - \dot{x}(5^-) = 1/2$. Since $\dot{x}(5^-) = 0$, the appropriate initial conditions immediately after the impulse force are $x(5^+) = 0$ and $\dot{x}(5^+) = 1/2$. This result can be confirmed using (4.14).

Chapter 5

Series solutions of homogeneous linear second-order differential equations

Reference: Boyce and DiPrima, Chapter 5

We consider the homogeneous linear second-order differential equation for $y = y(x)$:

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{5.1}$$

where $P(x)$, $Q(x)$ and $R(x)$ are polynomials or convergent power series around $x = x_0$, with no common polynomial factors that could be divided out. The value $x = x_0$ is called an *ordinary point* of (5.1) if $P(x_0) \neq 0$, and is called a singular point if $P(x_0) = 0$. Singular points will later be further classified as *regular singular points* and *irregular singular points*. Our goal is to find two independent solutions of (5.1).

5.1 Ordinary points

If x_0 is an ordinary point of (5.1), then it is possible to determine two power series (i.e., Taylor series) solutions for $y = y(x)$ centered at $x = x_0$. We illustrate the method of solution by solving two examples, with $x_0 = 0$.

Example: Find the general solution of $y'' + y = 0$.

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By now, you should know that the general solution is $y(x) = a_0 \cos x + a_1 \sin x$, with a_0 and a_1 constants. To find a power series solution about the point $x_0 = 0$, we write

$$y(x) = \sum_{n=0}^{\infty} a_n x^n;$$

and upon differentiating term-by-term

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting the power series for y and its derivatives into the differential equation to be solved, we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0. \tag{5.2}$$

The power-series solution method requires combining the two sums on the left-hand-side of (5.2) into a single power series in x. To shift the exponent of x^{n-2} in the first sum upward by two to obtain x^n, we need to shift the summation index downward by two; that is,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

We can then combine the two sums in (5.2) to obtain

$$\sum_{n=0}^{\infty} \Big((n+2)(n+1)a_{n+2} + a_n \Big) x^n = 0. \tag{5.3}$$

For (5.3) to be satisfied, the coefficient of each power of x must vanish separately. (This can be proved by setting $x = 0$ after successive differentiation.) We therefore obtain the *recurrence relation*

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, \ldots.$$

We observe that even and odd coefficients decouple. We thus obtain two independent sequences starting with first term a_0 or a_1. Developing these sequences, we have for the sequence beginning with a_0:

$$\begin{aligned}
&a_0, \\
&a_2 = -\frac{1}{2}a_0, \\
&a_4 = -\frac{1}{4\cdot 3}a_2 = \frac{1}{4\cdot 3\cdot 2}a_0, \\
&a_6 = -\frac{1}{6\cdot 5}a_4 = -\frac{1}{6!}a_0;
\end{aligned}$$

and the general coefficient in this sequence for $n = 0, 1, 2, \ldots$ is

$$a_{2n} = \frac{(-1)^n}{(2n)!}a_0.$$

Also, for the sequence beginning with a_1:

$$\begin{aligned}
&a_1, \\
&a_3 = -\frac{1}{3\cdot 2}a_1, \\
&a_5 = -\frac{1}{5\cdot 4}a_3 = \frac{1}{5\cdot 4\cdot 3\cdot 2}a_1, \\
&a_7 = -\frac{1}{7\cdot 6}a_5 = -\frac{1}{7!}a_1;
\end{aligned}$$

and the general coefficient in this sequence for $n = 0, 1, 2, \dots$ is

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1.$$

Using the principle of superposition, the general solution is therefore

$$\begin{aligned} y(x) &= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\ &= a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \\ &= a_0 \cos x + a_1 \sin x, \end{aligned}$$

as expected.

In our next example, we will solve the *Airy's Equation*. This differential equation arises in the study of optics, fluid mechanics, and quantum mechanics.

Example: Find the general solution of $y'' - xy = 0$.

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With

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

the differential equation becomes

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0. \tag{5.4}$$

We shift the first sum to x^{n+1} by shifting the exponent up by three, i.e.,

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=-1}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1}.$$

When combining the two sums in (5.4), we separate out the extra $n = -1$ term in the first sum given by $2a_2$. Therefore, (5.4) becomes

$$2a_2 + \sum_{n=0}^{\infty} \Big((n+3)(n+2) a_{n+3} - a_n \Big) x^{n+1} = 0. \tag{5.5}$$

Setting coefficients of powers of x to zero, we first find $a_2 = 0$, and then obtain the recursion relation

$$a_{n+3} = \frac{1}{(n+3)(n+2)} a_n. \tag{5.6}$$

Three sequences of coefficients—those starting with either a_0, a_1 or a_2—decouple. In particular the three sequences are

$$\begin{aligned} &a_0, a_3, a_6, a_9, \dots; \\ &a_1, a_4, a_7, a_{10}, \dots; \\ &a_2, a_5, a_8, a_{11} \dots. \end{aligned}$$

Since $a_2 = 0$, we find immediately for the last sequence

$$a_2 = a_5 = a_8 = a_{11} = \cdots = 0.$$

We compute the first four nonzero terms in the power series with coefficients corresponding to the first two sequences. Starting with a_0, we have

$$\begin{aligned} &a_0, \\ a_3 &= \frac{1}{3 \cdot 2} a_0, \\ a_6 &= \frac{1}{6 \cdot 5 \cdot 3 \cdot 2} a_0, \\ a_9 &= \frac{1}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} a_0; \end{aligned}$$

and starting with a_1,

$$\begin{aligned} &a_1, \\ a_4 &= \frac{1}{4 \cdot 3} a_1, \\ a_7 &= \frac{1}{7 \cdot 6 \cdot 4 \cdot 3} a_1, \\ a_{10} &= \frac{1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3} a_1. \end{aligned}$$

The general solution for $y = y(x)$, can therefore be written as

$$\begin{aligned} y(x) &= a_0 \left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12960} + \ldots\right) + a_1 \left(x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45360} + \ldots\right) \\ &= a_0 y_0(x) + a_1 y_1(x). \end{aligned}$$

Suppose we would like to graph the solutions $y = y_0(x)$ and $y = y_1(x)$ versus x by solving the differential equation $y'' - xy = 0$ numerically. What initial conditions should we use? Clearly, $y = y_0(x)$ solves the ode with initial values $y(0) = 1$ and $y'(0) = 0$, while $y = y_1(x)$ solves the ode with initial values $y(0) = 0$ and $y'(0) = 1$.

The numerical solutions, obtained using MATLAB, are shown in Fig. 5.1. Note that the solutions oscillate for negative x and grow exponentially for positive x. This can be understood by recalling that $y'' + y = 0$ has oscillatory sine and cosine solutions and $y'' - y = 0$ has exponential hyperbolic sine and cosine solutions.

5.2 Regular singular points: Cauchy-Euler equations

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The value $x = x_0$ is called a regular singular point of the ode

$$(x - x_0)^2 y'' + p(x)(x - x_0)y' + q(x)y = 0, \tag{5.7}$$

if $p(x)$ and $q(x)$ have convergent Taylor series about $x = x_0$, i.e., $p(x)$ and $q(x)$ can be written as a power-series in $(x - x_0)$:

$$\begin{aligned} p(x) &= p_0 + p_1(x - x_0) + p_2(x - x_0)^2 + \ldots, \\ q(x) &= q_0 + q_1(x - x_0) + q_2(x - x_0)^2 + \ldots, \end{aligned}$$

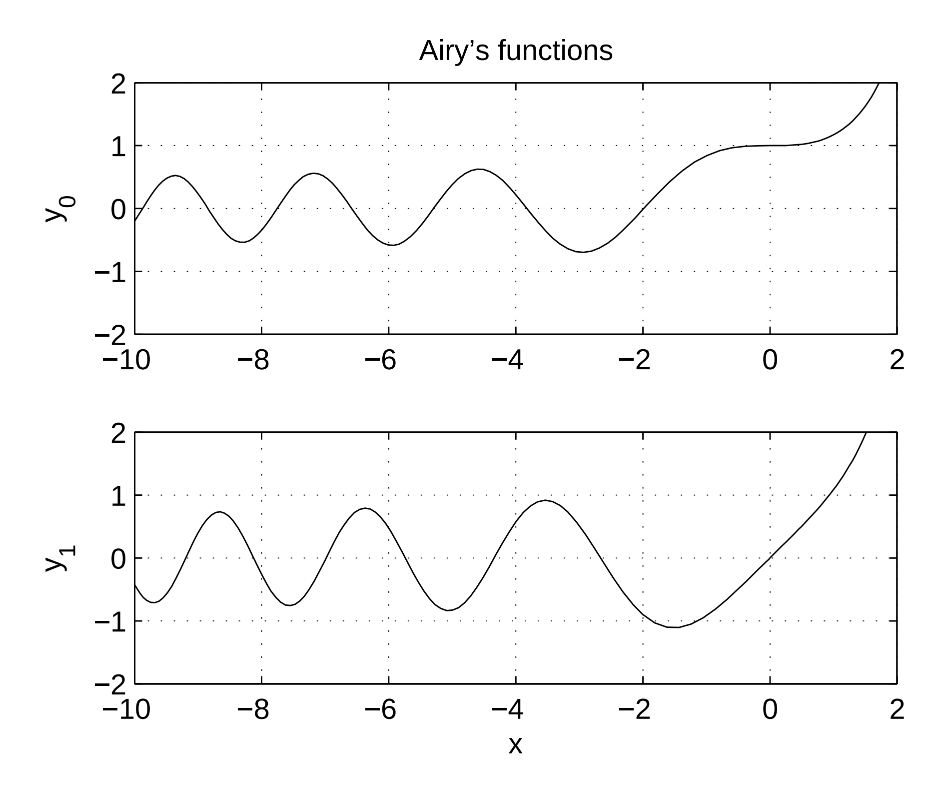


Figure 5.1: *Numerical solution of Airy's equation.*

with p_n and q_n constants, and $q_0 \neq 0$ so that $(x - x_0)$ is not a common factor of the coefficients. Any point $x = x_0$ that is not an ordinary point or a regular singular point is called an irregular singular point. Many important differential equations of physical interest have regular singular points, and their solutions go by the generic name of special functions, with specific names associated with now famous mathematicians like Bessel, Legendre, Hermite, Laguerre and Chebyshev.

Here, we will only consider the simplest ode with a regular singular point at $x = 0$. This ode is called a *Cauchy-Euler equation*, and has the form

$$x^2y'' + \alpha xy' + \beta y = 0, \tag{5.8}$$

with α and β constants. Note that (5.7) reduces to a Cauchy-Euler equation (about $x = x_0$) when one considers only the leading-order term in the Taylor series expansion of the functions $p(x)$ and $q(x)$. In fact, taking $p(x) = p_0$ and $q(x) = q_0$ and solving the associated Cauchy-Euler equation results in at least one of the leading-order solutions to the more general ode (5.7). Often, this is sufficient to obtain initial conditions for numerical solution of the full ode. Students wishing to learn how to find the general solution of (5.7) can consult Boyce & DiPrima.

An appropriate ansatz for (5.8) is $y = x^r$, when $x > 0$ and $y = (-x)^r$ when $x < 0$, (or more generally, $y = |x|^r$ for all x), with r constant. After substitution into (5.8), we obtain for both positive and negative x

$$r(r-1)|x|^r + \alpha r|x|^r + \beta|x|^r = 0,$$

and we observe that our ansatz is rewarded by cancelation of $|x|^r$. We thus obtain

the following quadratic equation for r:

$$r^2 + (\alpha - 1)r + \beta = 0, \tag{5.9}$$

which can be solved using the quadratic formula. Three cases immediately appear: (i) real distinct roots, (ii) complex conjugate roots, (iii) repeated roots. Students may recall being in a similar situation when solving the second-order linear homogeneous ode with constant coefficients. Indeed, it is possible to directly transform the Cauchy-Euler equation into an equation with constant coefficients so that our previous results can be used.

The idea is to change variables so that the power law ansatz $y = x^r$ becomes an exponential ansatz. For $x > 0$, if we let $x = e^\xi$ and $y(x) = Y(\xi)$, then the ansatz $y(x) = x^r$ becomes the ansatz $Y(\xi) = e^{r\xi}$, appropriate if $Y(\xi)$ satisfies a constant coefficient ode. If $x < 0$, then the appropriate transformation is $x = -e^\xi$, since $e^\xi > 0$. We need only consider $x > 0$ here and subsequently generalize our result by replacing x everywhere by its absolute value.

We thus transform the differential equation (5.8) for $y = y(x)$ into a differential equation for $Y = Y(\xi)$, using $x = e^\xi$, or equivalently, $\xi = \ln x$. By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dY}{d\xi}\frac{d\xi}{dx} \\ &= \frac{1}{x}\frac{dY}{d\xi} \\ &= e^{-\xi}\frac{dY}{d\xi},\end{aligned}$$

so that symbolically,

$$\frac{d}{dx} = e^{-\xi}\frac{d}{d\xi}.$$

The second derivative transforms as

$$\begin{aligned}\frac{d^2y}{dx^2} &= e^{-\xi}\frac{d}{d\xi}\left(e^{-\xi}\frac{dY}{d\xi}\right) \\ &= e^{-2\xi}\left(\frac{d^2Y}{d\xi^2} - \frac{dY}{d\xi}\right).\end{aligned}$$

Upon substitution of the derivatives of y into (5.8), and using $x = e^\xi$, we obtain

$$\begin{aligned}e^{2\xi}\left(e^{-2\xi}\left(Y'' - Y'\right)\right) + \alpha e^{\xi}\left(e^{-\xi}Y'\right) + \beta Y = Y'' + (\alpha - 1)Y' + \beta Y \\ = 0.\end{aligned}$$

As expected, the ode for $Y = Y(\xi)$ has constant coefficients, and with $Y = e^{r\xi}$, the characteristic equation for r is given by (5.9). We now directly transfer previous results obtained for the constant coefficient second-order linear homogeneous ode.

5.2.1 Distinct real roots

This simplest case needs no transformation. If $(\alpha - 1)^2 - 4\beta > 0$, then with $r_\pm$ the real roots of (5.9), the general solution is

$$y(x) = c_1|x|^{r_+} + c_2|x|^{r_-}.$$

5.2.2 Distinct complex-conjugate roots

If $(\alpha - 1)^2 - 4\beta < 0$, we can write the complex roots of (5.9) as $r_\pm = \lambda \pm i\mu$. Recall the general solution for $Y = Y(\xi)$ is given by

$$Y(\xi) = e^{\lambda\xi} \left(A \cos \mu\xi + B \sin \mu\xi\right);$$

and upon transformation, and replacing x by $|x|$,

$$y(x) = |x|^\lambda \big(A \cos \left(\mu \ln |x|\right) + B \sin \left(\mu \ln |x|\right)\big).$$

5.2.3 Repeated roots

If $(\alpha - 1)^2 - 4\beta = 0$, there is one real root r of (5.9). The general solution for Y is

$$Y(\xi) = e^{r\xi} \left(c_1 + c_2\xi\right),$$

yielding

$$y(x) = |x|^r \left(c_1 + c_2 \ln |x|\right).$$

We now give examples illustrating these three cases.

Example: Solve $2x^2y'' + 3xy' - y = 0$ for $0 \leq x \leq 1$ with two-point boundary condition $y(0) = 0$ and $y(1) = 1$.

Since $x > 0$, we try $y = x^r$ and obtain the characteristic equation

$$\begin{aligned} 0 &= 2r(r-1) + 3r - 1 \\ &= 2r^2 + r - 1 \\ &= (2r-1)(r+1). \end{aligned}$$

Since the characteristic equation has two real roots, the general solution is given by

$$y(x) = c_1 x^{\frac{1}{2}} + c_2 x^{-1}.$$

We now encounter for the first time two-point boundary conditions, which can be used to determine the coefficients c_1 and c_2. Since y(0)=0, we must have $c_2 = 0$. Applying the remaining condition $y(1) = 1$, we obtain the unique solution

$$y(x) = \sqrt{x}.$$

Note that $x = 0$ is called a *singular point* of the ode since the general solution is singular at $x = 0$ when $c_2 \neq 0$. Our boundary condition imposes that $y(x)$ is finite at $x = 0$ removing the singular solution. Nevertheless, y' remains singular at $x = 0$. Indeed, this is why we imposed a two-point boundary condition rather than specifying the value of $y'(0)$ (which is infinite).

Example: Solve $x^2y'' + xy' + \pi^2 y = 0$ with two-point boundary condition $y(1) = 1$ and $y(\sqrt{e}) = 1$.

With the ansatz $y = x^r$, we obtain

$$\begin{aligned} 0 &= r(r-1) + r + \pi^2 \\ &= r^2 + \pi^2, \end{aligned}$$

so that $r = \pm i\pi$. Therefore, with $\xi = \ln x$, we have $Y(\xi) = A\cos\pi\xi + B\sin\pi\xi$, and the general solution for $y(x)$ is

$$y(x) = A\cos(\pi\ln x) + B\sin(\pi\ln x).$$

The first boundary condition $y(1) = 1$ yields $A = 1$. The second boundary condition $y(\sqrt{e}) = 1$ yields $B = 1$.

Example: Solve $x^2y'' + 5xy' + 4y = 0$ **with two-point boundary condition** $y(1) = 0$ **and** $y(e) = 1$**.**

With the ansatz $y = x^r$, we obtain

$$\begin{aligned} 0 &= r(r-1) + 5r + 4 \\ &= r^2 + 4r + 4 \\ &= (r+2)^2, \end{aligned}$$

so that there is a repeated root $r = -2$. With $\xi = \ln x$, we have $Y(\xi) = (c_1 + c_2\xi)e^{-2\xi}$, so that the general solution is

$$y(x) = \frac{c_1 + c_2\ln x}{x^2}.$$

The first boundary condition $y(1) = 0$ yields $c_1 = 0$. The second boundary condition $y(e) = 1$ yields $c_2 = e^2$. The solution is therefore

$$y(x) = \frac{e^2\ln x}{x^2}.$$

Chapter 6

Systems of linear first-order equations

Reference: Boyce and DiPrima, Chapter 7

Systems of coupled linear differential equations can result, for example, from linear stability analyses of nonlinear equations, and from normal mode analyses of coupled oscillators. We will first consider the simplest case of a system of two coupled homogeneous linear first-order equations with constant coefficients. These two first-order equations are in fact equivalent to a single second-order equation, and the methods of Chapter 3 could be used for solution. Nevertheless, viewing the problem as a system of first-order equations introduces the important concept of the phase space, and can easily be generalized to higher-order linear systems. We will then discuss the physical problem of two coupled oscillators.

6.1 Matrices, determinants and the eigenvalue problem

View a lecture on matrix addition and multiplication on YouTube
View a lecture on determinants on YouTube

We begin by reviewing some basic matrix algebra. A matrix with n rows and m columns is called an n-by-m matrix. Here, we need only consider the simple case of two-by-two matrices.

A two-by-two matrix A, with two rows and two columns, can be written as

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The first row has elements a and b, the second row has elements c and d. The first column has elements a and c; the second column has elements b and d.

Matrices can be added and multiplied. Matrices can be added if they have the same dimension, and addition proceeds element by element, following

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}.$$

Matrices can be multiplied if the number of columns of the left matrix equals the number of rows of the right matrix. A particular element in the resulting product matrix, say in row k and column l, is obtained by multiplying and summing the elements in row k of the left matrix with the elements in column l of the right matrix. For example, a two-by-two matrix can multiply a two-by-one column vector as follows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

The first row of the left matrix is multiplied against and summed with the first (and only) column of the right matrix to obtain the element in the first row and first column of the product matrix, and so on for the element in the second row and first column. The product of two two-by-two matrices is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

A system of linear algebraic equations is easily represented in matrix form. For instance, with a_{ij} and b_i given numbers, and x_i unknowns, the (two-by-two) system of equations given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1, \\ a_{21}x_1 + a_{22}x_2 &= b_2, \end{aligned}$$

can be written in matrix form as

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{6.1}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{6.2}$$

When $\mathbf{b} = 0$, we say that the system of equations given by (6.1) is homogeneous. We now ask the following question: When does there exist a nontrivial (not identically zero) solution for $\mathbf{x}$ when the linear system is homogeneous? For the simplest case of a two-by-two matrix, we can try and solve directly the homogeneous linear system of equations given by

$$\begin{aligned} ax_1 + bx_2 &= 0, \\ cx_1 + dx_2 &= 0. \end{aligned} \tag{6.3}$$

Multiplying the first equation by d and the second by b, and subtracting the second equation from the first, results in

$$(ad - bc)x_1 = 0.$$

Similarly, multiplying the first equation by c and the second by a, and subtracting the first equation from the second, results in

$$(ad - bc)x_2 = 0.$$

Therefore, a nontrivial solution of (6.3) for a two-by-two matrix exists only if $ad - bc = 0$. If we define the determinant of the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.4}$$

to be $\det \mathbf{A} = ad - bc$, then we say that a nontrivial solution to (6.3) exists provided $\det \mathbf{A} = 0$.

The same calculation may be repeated for a 3×3 matrix. If

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

then there exists a nontrivial solution to $\mathbf{Ax} = 0$ provided $\det \mathbf{A} = 0$, where $\det \mathbf{A} = a(ei - fh) - b(di - fg) + c(dh - eg)$. The definition of the determinant can be further generalized to any $n \times n$ matrix, and is typically taught in a first course on linear algebra.

We now consider the eigenvalue problem. For $\mathbf{A}$ an $n \times n$ matrix and $\mathbf{v}$ an $n \times 1$ column vector, the eigenvalue problem solves the equation

$$\mathbf{Av} = \lambda \mathbf{v} \tag{6.5}$$

for eigenvalues λ_i and corresponding eigenvectors $\mathbf{v}_i$. We rewrite the eigenvalue equation (6.5) as

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = 0, \tag{6.6}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix, that is, the matrix with ones on the diagonal and zeros everywhere else. A nontrivial solution of (6.6) exists provided

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0. \tag{6.7}$$

Equation (6.7) is an n-th order polynomial equation in λ, and is called the *characteristic equation* of $\mathbf{A}$. The characteristic equation can be solved for the eigenvalues, and for each eigenvalue, a corresponding eigenvector can be determined directly from (6.5).

We can demonstrate how to find the eigenvalues and eigenvectors of the 2×2 matrix given by (6.4). We have

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda \mathbf{I}) \\ &= \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc). \end{aligned}$$

This characteristic equation can be more generally written as

$$\lambda^2 - \operatorname{Tr} \mathbf{A}\, \lambda + \det \mathbf{A} = 0, \tag{6.8}$$

where $\operatorname{Tr} \mathbf{A}$ is the trace, or sum of the diagonal elements, of the matrix $\mathbf{A}$. If λ is an eigenvalue of $\mathbf{A}$, then the corresponding eigenvector $\mathbf{v}$ may be found by solving

$$\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0,$$

where the equation of the second row will always be a multiple of the equation of the first row. The eigenvector $\mathbf{v}$ has arbitrary normalization, and we may always choose for convenience $v_1 = 1$. The equation from the first row is

$$(a - \lambda)v_1 + bv_2 = 0,$$

and with $v_1 = 1$, we find $v_2 = (\lambda - a)/b$.

In the next section, we will see several examples of an eigenvector analysis.

6.2 Two coupled homogeneous linear first-order differential equations

We now consider the general system of differential equations given by

$$\dot{x}_1 = ax_1 + bx_2, \quad \dot{x}_2 = cx_1 + dx_2, \tag{6.9}$$

which can be written using vector notation as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}. \tag{6.10}$$

Before solving this system of odes using matrix techniques, I first want to show that we could actually solve these equations by converting the system into a single second-order equation. We take the derivative of the first equation and use both equations to write

$$\begin{aligned}\ddot{x}_1 &= a\dot{x}_1 + b\dot{x}_2 \\ &= a\dot{x}_1 + b(cx_1 + dx_2) \\ &= a\dot{x}_1 + bcx_1 + d(\dot{x}_1 - ax_1) \\ &= (a+d)\dot{x}_1 - (ad - bc)x_1.\end{aligned}$$

The system of two first-order equations therefore becomes the following second-order equation:

$$\ddot{x}_1 - (a+d)\dot{x}_1 + (ad - bc)x_1 = 0.$$

If we had taken the derivative of the second equation instead, we would have obtained the identical equation for x_2:

$$\ddot{x}_2 - (a+d)\dot{x}_2 + (ad - bc)x_2 = 0.$$

In general, a system of n first-order linear homogeneous equations can be converted into an equivalent n-th order linear homogeneous equation. Numerical methods usually require the conversion in reverse; that is, a conversion of an n-th order equation into a system of n first-order equations.

With the ansatz $x_1 = e^{\lambda t}$ or $x_2 = e^{\lambda t}$, the second-order odes have the characteristic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

This is identical to the characteristic equation obtained for the matrix $\mathbf{A}$ from an eigenvalue analysis.

We will see that (6.9) can in fact be solved by considering the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. We will demonstrate the solution for three separate cases: (i) eigenvalues of $\mathbf{A}$ are real and there are two linearly independent eigenvectors; (ii) eigenvalues of $\mathbf{A}$ are complex conjugates, and; (iii) $\mathbf{A}$ has only one linearly independent eigenvector. These three cases are analogous to the cases considered previously when solving the homogeneous linear constant-coefficient second-order equation.

6.2.1 Distinct real eigenvalues

We illustrate the solution method by example.

Example: Find the general solution of $\dot{x}_1 = x_1 + x_2$, $\dot{x}_2 = 4x_1 + x_2$.

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The equation to be solved may be rewritten in matrix form as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

or using vector notation, written as (6.10). We take as our ansatz $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$, where $\mathbf{v}$ and λ are independent of t. Upon substitution into (6.10), we obtain

$$\lambda \mathbf{v} e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t};$$

and upon cancelation of the exponential, we obtain the eigenvalue problem

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}. \tag{6.11}$$

Finding the characteristic equation using (6.8), we have

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda \mathbf{I}) \\ &= \lambda^2 - 2\lambda - 3 \\ &= (\lambda - 3)(\lambda + 1). \end{aligned}$$

Therefore, the two eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$.

To determine the corresponding eigenvectors, we substitute the eigenvalues successively into

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = 0. \tag{6.12}$$

We will write the corresponding eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$ using the matrix notation

$$\begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix},$$

where the components of $\mathbf{v_1}$ and $\mathbf{v_2}$ are written with subscripts corresponding to the first and second columns of a 2×2 matrix.

For $\lambda_1 = 3$, and unknown eigenvector $\mathbf{v}_1$, we have from (6.12)

$$\begin{aligned} -2v_{11} + v_{21} &= 0, \\ 4v_{11} - 2v_{21} &= 0. \end{aligned}$$

Clearly, the second equation is just the first equation multiplied by -2, so only one equation is linearly independent. This will always be true, so for the 2×2 case we need only consider the first row of the matrix. The first eigenvector therefore satisfies $v_{21} = 2v_{11}$. Recall that an eigenvector is only unique up to multiplication by a constant: we may therefore take $v_{11} = 1$ for convenience.

For $\lambda_2 = -1$, and eigenvector $\mathbf{v}_2 = (v_{12}, v_{22})^T$, we have from (6.12)

$$2v_{12} + v_{22} = 0,$$

so that $v_{22} = -2v_{12}$. Here, we take $v_{12} = 1$.

Therefore, our eigenvalues and eigenvectors are given by

$$\lambda_1 = 3,\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad \lambda_2 = -1,\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

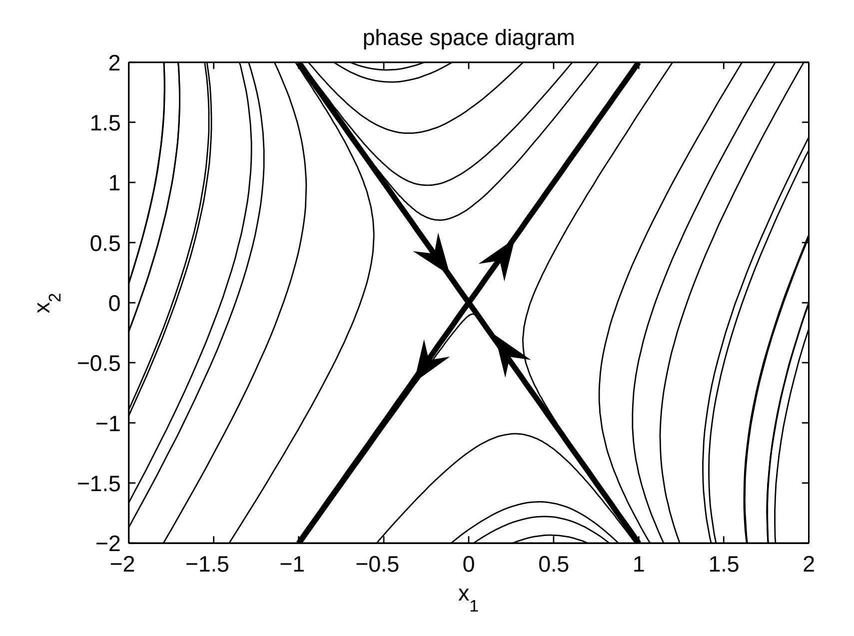


Figure 6.1: *Phase space diagram for example with two real eigenvalues of opposite sign.*

Using the principle of superposition, the general solution to the ode is therefore

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t},$$

or explicitly writing out the components,

$$\begin{aligned} x_1(t) &= c_1 e^{3t} + c_2 e^{-t}, \\ x_2(t) &= 2c_1 e^{3t} - 2c_2 e^{-t}. \end{aligned}$$

We can obtain a new perspective on the solution by drawing a phase-space diagram, shown in Fig. 6.1, with "x-axis" x_1 and "y-axis" x_2. Each curve corresponds to a different initial condition, and represents the trajectory of a particle with velocity given by the differential equation. The dark lines represent trajectories along the direction of the eigenvectors. If $c_2 = 0$, the motion is along the eigenvector $\mathbf{v_1}$ with $x_2 = 2x_1$ and the motion with increasing time is away from the origin (arrows pointing out) since the eigenvalue $\lambda_1 = 3 > 0$. If $c_1 = 0$, the motion is along the eigenvector $\mathbf{v_2}$ with $x_2 = -2x_1$ and motion is towards the origin (arrows pointing in) since the eigenvalue $\lambda_2 = -1 < 0$. When the eigenvalues are real and of opposite signs, the origin is called a *saddle point*. Almost all trajectories (with the exception of those with initial conditions exactly satisfying $x_2(0) = -2x_1(0)$) eventually move away from the origin as t increases.

Example: Find the general solution of $\dot{x}_1 = -3x_1 + \sqrt{2}x_2$, $\dot{x}_2 = \sqrt{2}x_1 - 2x_2$.

The equations in matrix form are

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

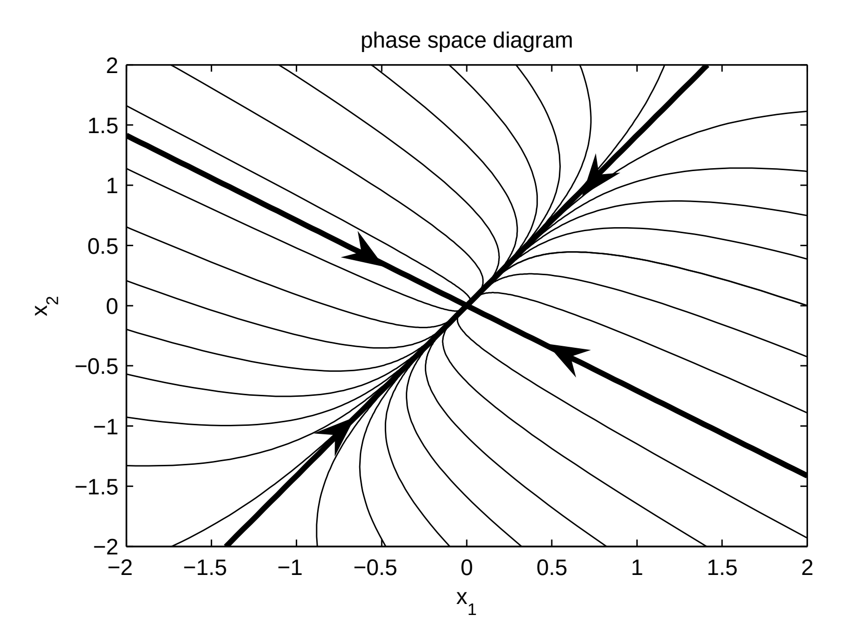


Figure 6.2: *Phase space diagram for example with two real eigenvalues of same sign.*

The ansatz $\mathbf{x} = \mathbf{v}e^{\lambda t}$ leads to the eigenvalue problem $\mathbf{Av} = \lambda\mathbf{v}$, with $\mathbf{A}$ the matrix above. The eigenvalues are determined from

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda\mathbf{I}) \\ &= \lambda^2 + 5\lambda + 4 \\ &= (\lambda + 4)(\lambda + 1). \end{aligned}$$

Therefore, the eigenvalues of $\mathbf{A}$ are $\lambda_1 = -4$, $\lambda_2 = -1$. Proceeding to determine the associated eigenvectors, for $\lambda_1 = -4$,

$$v_{11} + \sqrt{2}v_{21} = 0;$$

and for $\lambda_2 = -1$,

$$-2v_{12} + \sqrt{2}v_{22} = 0.$$

Taking the normalization $v_{11} = 1$ and $v_{12} = 1$, we obtain for the eigenvalues and associated eigenvectors

$$\lambda_1 = -4,\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ -\sqrt{2}/2 \end{pmatrix}; \qquad \lambda_2 = -1,\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

The general solution to the ode is therefore

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -\sqrt{2}/2 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}.$$

We show the phase space plot in Fig. 6.2. If $c_2 = 0$, the motion is along the eigenvector $\mathbf{v_1}$ with $x_2 = -(\sqrt{2}/2)x_1$ with eigenvalue $\lambda_1 = -4 < 0$. If $c_1 = 0$, the motion is along the eigenvector $\mathbf{v_2}$ with $x_2 = \sqrt{2}x_1$ with eigenvalue $\lambda_2 = -1 < 0$. When the eigenvalues are real and have the same sign, the origin is called a *node*.

A node may be attracting or repelling depending on whether the eigenvalues are both negative (as is the case here) or positive. Observe that the trajectories collapse onto the $\mathbf{v}_2$ eigenvector since $\lambda_1 < \lambda_2 < 0$ and decay is more rapid along the $\mathbf{v}_1$ direction.

6.2.2 Distinct complex-conjugate eigenvalues

Example: Find the general solution of $\dot{x}_1 = -\frac{1}{2}x_1 + x_2$, $\dot{x}_2 = -x_1 - \frac{1}{2}x_2$.

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The equations in matrix form are

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The ansatz $\mathbf{x} = \mathbf{v}e^{\lambda t}$ leads to the equation

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda\mathbf{I}) \\ &= \lambda^2 + \lambda + \frac{5}{4}. \end{aligned}$$

Therefore, $\lambda = -1/2 \pm i$; and we observe that the eigenvalues occur as a complex conjugate pair. We will denote the two eigenvalues as

$$\lambda = -\frac{1}{2} + i \quad \text{and} \quad \bar{\lambda} = -\frac{1}{2} - i.$$

Now, for $\mathbf{A}$ a real matrix, if $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, then $\mathbf{A}\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$. Therefore, the eigenvectors also occur as a complex conjugate pair. The eigenvector $\mathbf{v}$ associated with eigenvalue λ satisfies $-iv_1 + v_2 = 0$, and normalizing with $v_1 = 1$, we have

$$\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

We have therefore determined two independent complex solutions to the ode, that is,

$$\mathbf{v}e^{\lambda t} \quad \text{and} \quad \bar{\mathbf{v}}e^{\bar{\lambda} t},$$

and we can form a linear combination of these two complex solutions to construct two independent real solutions. Namely, if the complex functions $z(t)$ and $\bar{z}(t)$ are written as

$$z(t) = \text{Re}\{z(t)\} + i\text{Im}\{z(t)\}, \quad \bar{z}(t) = \text{Re}\{z(t)\} - i\text{Im}\{z(t)\},$$

then two real functions can be constructed from the following linear combinations of z and $\bar{z}$:

$$\frac{z + \bar{z}}{2} = \text{Re}\{z(t)\} \quad \text{and} \quad \frac{z - \bar{z}}{2i} = \text{Im}\{z(t)\}.$$

Thus the two real vector functions that can be constructed from our two complex vector functions are

$$\begin{aligned} \text{Re}\{\mathbf{v}e^{\lambda t}\} &= \text{Re}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-\frac{1}{2}+i)t}\right\} \\ &= e^{-\frac{1}{2}t}\text{Re}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix}(\cos t + i\sin t)\right\} \\ &= e^{-\frac{1}{2}t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}; \end{aligned}$$

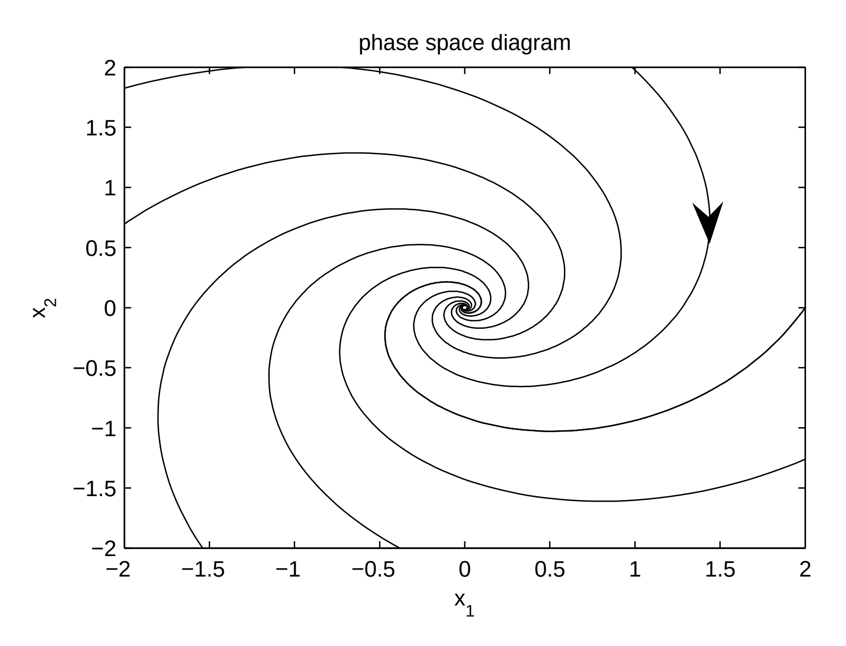


Figure 6.3: *Phase space diagram for example with complex conjugate eigenvalues.*

and

$$\begin{aligned}\text{Im}\{\mathbf{v}e^{\lambda t}\} &= e^{-\frac{1}{2}t}\text{Im}\left\{\begin{pmatrix}1\\ i\end{pmatrix}(\cos t + i\sin t)\right\}\\ &= e^{-\frac{1}{2}t}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}.\end{aligned}$$

Taking a linear superposition of these two real solutions yields the general solution to the ode, given by

$$\mathbf{x} = e^{-\frac{1}{2}t}\left(A\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + B\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}\right).$$

The corresponding phase space diagram is shown in Fig. 6.3. We say the origin is a *spiral point*. If the real part of the complex eigenvalue is negative, as is the case here, then solutions spiral into the origin. If the real part of the eigenvalue is positive, then solutions spiral out of the origin.

The direction of the spiral—here, it is clockwise—can be determined easily. If we examine the ode with $x_1 = 0$ and $x_2 = 1$, we see that $\dot{x}_1 = 1$ and $\dot{x}_2 = -1/2$. The trajectory at the point $(0, 1)$ is moving to the right and downward, and this is possible only if the spiral is clockwise. A counterclockwise trajectory would be moving to the left and downward.

6.2.3 Repeated eigenvalues with one eigenvector

Example: Find the general solution of $\dot{x}_1 = x_1 - x_2$, $\dot{x}_2 = x_1 + 3x_2$.

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The equations in matrix form are

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{6.13}$$

The ansatz $\mathbf{x} = \mathbf{v}e^{\lambda t}$ leads to the characteristic equation

$$\begin{aligned} 0 &= \det\left(\mathbf{A} - \lambda\mathbf{I}\right) \\ &= \lambda^2 - 4\lambda + 4 \\ &= (\lambda - 2)^2. \end{aligned}$$

Therefore, $\lambda = 2$ is a repeated eigenvalue. The associated eigenvector is found from $-v_1 - v_2 = 0$, or $v_2 = -v_1$; and normalizing with $v_1 = 1$, we have

$$\lambda = 2, \quad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We have thus found a single solution to the ode, given by

$$\mathbf{x_1}(t) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t},$$

and we need to find the missing second solution to be able to satisfy the initial conditions. An ansatz of t times the first solution is tempting, but will fail. Here, we will cheat and find the missing second solution by solving the equivalent second-order, homogeneous, constant-coefficient differential equation.

We already know that this second-order differential equation for $x_1(t)$ has a characteristic equation with a degenerate eigenvalue given by $\lambda = 2$. Therefore, the general solution for x_1 is given by

$$x_1(t) = (c_1 + tc_2)e^{2t}.$$

Since from the first differential equation, $x_2 = x_1 - \dot{x}_1$, we compute

$$\dot{x}_1 = \left(2c_1 + (1 + 2t)c_2\right)e^{2t},$$

so that

$$\begin{aligned} x_2 &= x_1 - \dot{x}_1 \\ &= (c_1 + tc_2)e^{2t} - \left(2c_1 + (1 + 2t)c_2\right)e^{2t} \\ &= -c_1e^{2t} + c_2(-1 - t)e^{2t}. \end{aligned}$$

Combining our results for x_1 and x_2, we have therefore found

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}t\right]e^{2t}.$$

Our missing linearly independent solution is thus determined to be

$$\mathbf{x}(t) = c_2\left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}t\right]e^{2t}. \tag{6.14}$$

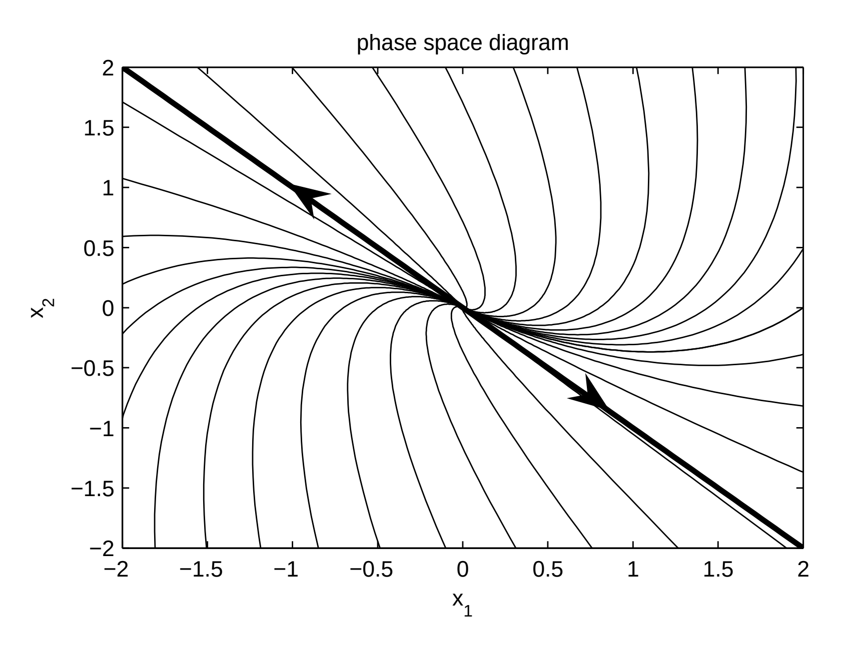


Figure 6.4: *Phase space diagram for example with only one eigenvector.*

The second term of (6.14) is just t times the first solution; however, this is not sufficient. Indeed, the correct ansatz to find the second solution directly is given by

$$\mathbf{x} = (\mathbf{w} + t\mathbf{v})\, e^{\lambda t}, \tag{6.15}$$

where λ and $\mathbf{v}$ is the eigenvalue and eigenvector of the first solution, and $\mathbf{w}$ is an unknown vector to be determined. To illustrate this direct method, we substitute (6.15) into $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, assuming $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. Canceling the exponential, we obtain

$$\mathbf{v} + \lambda\,(\mathbf{w} + t\mathbf{v}) = \mathbf{A}\mathbf{w} + \lambda t\mathbf{v}.$$

Further canceling the common term $\lambda t\mathbf{v}$ and rewriting yields

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{w} = \mathbf{v}. \tag{6.16}$$

If $\mathbf{A}$ has only a single linearly independent eigenvector $\mathbf{v}$, then (6.16) can be solved for $\mathbf{w}$ (otherwise, it cannot). Using $\mathbf{A}$, λ and $\mathbf{v}$ of our present example, (6.16) is the system of equations given by

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The first and second equation are the same, so that $w_2 = -(w_1 + 1)$. Therefore,

$$\begin{aligned} \mathbf{w} &= \begin{pmatrix} w_1 \\ -(w_1 + 1) \end{pmatrix} \\ &= w_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \end{aligned}$$

Notice that the first term repeats the first found solution, i.e., a constant times the eigenvector, and the second term is new. We therefore take $w_1 = 0$ and obtain

$$\mathbf{w} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

as before.

The phase space diagram for this ode is shown in Fig. 6.4. The dark line is the single eigenvector $\mathbf{v}$ of the matrix $\mathbf{A}$. When there is only a single eigenvector, the origin is called an *improper node*.

6.3 Normal modes

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We now consider an application of the eigenvector analysis to the coupled mass-spring system shown in Fig. 6.5. The position variables x_1 and x_2 are measured from the equilibrium positions of the masses. Hooke's law states that the spring force is linearly proportional to the extension length of the spring, measured from equilibrium. By considering the extension of the spring and the sign of the force, we write Newton's law $F = ma$ separately for each mass:

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - K(x_1 - x_2), \\ m\ddot{x}_2 &= -kx_2 - K(x_2 - x_1). \end{aligned}$$

Further rewriting by collecting terms proportional to x_1 and x_2 yields

$$\begin{aligned} m\ddot{x}_1 &= -(k+K)x_1 + Kx_2, \\ m\ddot{x}_2 &= Kx_1 - (k+K)x_2. \end{aligned}$$

The equations for the coupled mass-spring system form a system of two second-order linear homogeneous odes. In matrix form, $m\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, or explicitly,

$$m\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -(k+K) & K \\ K & -(k+K) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{6.17}$$

In analogy to a system of first-order equations, we try the ansatz $\mathbf{x} = \mathbf{v}e^{rt}$, and upon substitution into (6.17) we obtain the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, with $\lambda = mr^2$. The eigenvalues are determined by solving the characteristic equation

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda\mathbf{I}) \\ &= \begin{vmatrix} -(k+K) - \lambda & K \\ K & -(k+K) - \lambda \end{vmatrix} \\ &= (\lambda + k + K)^2 - K^2. \end{aligned}$$

The solution for λ is

$$\lambda = -k - K \pm K,$$

and the two eigenvalues are

$$\lambda_1 = -k, \quad \lambda_2 = -(k + 2K).$$

The corresponding values of r in our ansatz $\mathbf{x} = \mathbf{v}e^{rt}$, with $r = \pm\sqrt{\lambda/m}$, are

$$r_1 = i\sqrt{k/m}, \quad \bar{r}_1, \quad r_2 = i\sqrt{(k+2K)/m}, \quad \bar{r}_2.$$

Since the values of r are pure imaginary, we know that $x_1(t)$ and $x_2(t)$ will oscillate with angular frequencies $\omega_1 = \operatorname{Im}\{r_1\}$ and $\omega_2 = \operatorname{Im}\{r_2\}$, that is,

$$\omega_1 = \sqrt{k/m}, \quad \omega_2 = \sqrt{(k+2K)/m}.$$

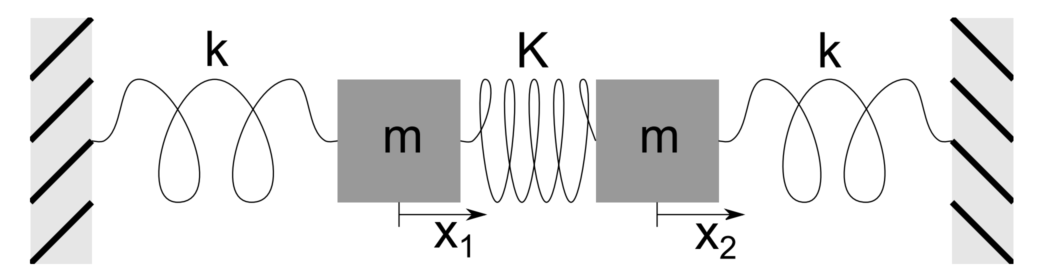


Figure 6.5: *Coupled harmonic oscillators.*

The positions of the oscillating masses in general contain time dependencies of the form $\sin \omega_1 t$, $\cos \omega_1 t$, and $\sin \omega_2 t$, $\cos \omega_2 t$.

It is of further interest to determine the eigenvectors, or so-called *normal modes* of oscillation, associated with the two distinct angular frequencies. With specific initial conditions proportional to an eigenvector, the mass will oscillate with a single frequency. The eigenvector with eigenvalue λ_1 satisfies

$$-Kv_{11} + Kv_{12} = 0,$$

so that $v_{11} = v_{12}$. The normal mode with frequency $\omega_1 = \sqrt{k/m}$ thus follows a motion where $x_1 = x_2$. Referring to Fig. 6.5, during this motion the center spring length does not change, which is why the frequency of oscillation is independent of K.

Next, we determine the eigenvector with eigenvalue λ_2:

$$Kv_{21} + Kv_{22} = 0,$$

so that $v_{21} = -v_{22}$. The normal mode with frequency $\omega_2 = \sqrt{(k + 2K)/m}$ thus follows a motion where $x_1 = -x_2$. Again referring to Fig. 6.5, during this motion the two equal masses symmetrically push or pull against each side of the middle spring.

A general solution for $\mathbf{x}(t)$ can be constructed from the eigenvalues and eigenvectors. Our ansatz was $\mathbf{x} = \mathbf{v}e^{rt}$, and for each of two eigenvectors $\mathbf{v}$, we have a pair of complex conjugate values for r. Accordingly, we first apply the principle of superposition to obtain four real solutions, and then apply the principle again to obtain the general solution. With $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{(k + 2K)/m}$, the general solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (A \cos \omega_1 t + B \sin \omega_1 t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (C \cos \omega_2 t + D \sin \omega_2 t),$$

where the now real constants A, B, C, and D can be determined from the four independent initial conditions, $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$, and $\dot{x}_2(0)$.

Chapter 7

Nonlinear differential equations and bifurcation theory

Reference: Strogatz, Sections 2.2, 2.4, 3.1, 3.2, 3.4, 6.3, 6.4, 8.2

We now turn our attention to nonlinear differential equations. In particular, we study how small changes in the parameters of a system can result in qualitative changes in the dynamics. These qualitative changes in the dynamics are called bifurcations. To understand bifurcations, we first need to understand the concepts of fixed points and stability.

7.1 Fixed points and stability

7.1.1 One dimension

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Consider the one-dimensional differential equation for $x = x(t)$ given by

$$\dot{x} = f(x). \tag{7.1}$$

We say that x_* is a *fixed point*, or *equilibrium point*, of (7.1) if $f(x_*) = 0$. At the fixed point, $\dot{x} = 0$. The terminology *fixed point* is used since the solution to (7.1) with initial condition $x(0) = x_*$ is $x(t) = x_*$ for all time t.

A fixed point, however, can be stable or unstable. A fixed point is said to be *stable* if a small perturbation of the solution from the fixed point decays in time; it is said to be *unstable* if a small perturbation grows in time. We can determine stability by a *linear analysis*. Let $x = x_* + \epsilon(t)$, where ϵ represents a small perturbation of the solution from the fixed point x_*. Because x_* is a constant, $\dot{x} = \dot{\epsilon}$; and because x_* is a fixed point, $f(x_*) = 0$. Taylor series expanding about $\epsilon = 0$, we have

$$\begin{aligned}
\dot{\epsilon} &= f(x_* + \epsilon) \\
&= f(x_*) + \epsilon f'(x_*) + \dots \\
&= \epsilon f'(x_*) + \dots .
\end{aligned}$$

The omitted terms in the Taylor series expansion are proportional to ϵ^2, and can be made negligible over a short time interval with respect to the kept term, proportional to ϵ, by taking $\epsilon(0)$ sufficiently small. Therefore, at least over short times, the differential equation to be considered, $\dot{\epsilon} = f'(x_*)\epsilon$, is linear and has by now the familiar solution

$$\epsilon(t) = \epsilon(0)e^{f'(x_*)t}.$$

The perturbation of the fixed point solution $x(t) = x_*$ thus decays exponentially if $f'(x_*) < 0$, and we say the fixed point is stable. If $f'(x_*) > 0$, the perturbation grows exponentially and we say the fixed point is unstable. If $f'(x_*) = 0$, we say the fixed point is marginally stable, and the next higher-order term in the Taylor series expansion must be considered.

Example: Find all the fixed points of the logistic equation $\dot{x} = x(1-x)$ and determine their stability.

There are two fixed points at which $\dot{x} = 0$, given by $x_* = 0$ and $x_* = 1$. Stability of these equilibrium points may be determined by considering the derivative of $f(x) = x(1-x)$. We have $f'(x) = 1-2x$. Therefore, $f'(0) = 1 > 0$ so that $x_* = 0$ is an unstable fixed point, and $f'(1) = -1 < 0$ so that $x_* = 1$ is a stable fixed point. Indeed, we have previously found that all solutions approach the stable fixed point asymptotically.

7.1.2 Two dimensions

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The idea of fixed points and stability can be extended to higher-order systems of odes. Here, we consider a two-dimensional system and will need to make use of the two-dimensional Taylor series expansion of a function $F(x,y)$ about the origin. In general, the Taylor series of $F(x,y)$ is given by

$$F(x,y) = F + x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + \frac{1}{2}\left(x^2\frac{\partial^2 F}{\partial x^2} + 2xy\frac{\partial^2 F}{\partial x \partial y} + y^2\frac{\partial^2 F}{\partial y^2}\right) + \dots,$$

where the function F and all of its partial derivatives on the right-hand-side are evaluated at the origin. Note that the Taylor series is constructed so that all partial derivatives of the left-hand-side match those of the right-hand-side at the origin.

We now consider the two-dimensional system given by

$$\dot{x} = f(x,y), \quad \dot{y} = g(x,y). \tag{7.2}$$

The point (x_*, y_*) is said to be a fixed point of (7.2) if $f(x_*, y_*) = 0$ and $g(x_*, y_*) = 0$. Again, the local stability of a fixed point can be determined by a linear analysis. We let $x(t) = x_* + \epsilon(t)$ and $y(t) = y_* + \delta(t)$, where ϵ and δ are small independent perturbations from the fixed point. Making use of the two dimensional Taylor series of $f(x,y)$ and $g(x,y)$ about the fixed point, or equivalently about $(\epsilon, \delta) = (0,0)$, we have

$$\begin{aligned}
\dot{\epsilon} &= f(x_* + \epsilon, y_* + \delta) \\
&= f + \epsilon\frac{\partial f}{\partial x} + \delta\frac{\partial f}{\partial y} + \dots \\
&= \epsilon\frac{\partial f}{\partial x} + \delta\frac{\partial f}{\partial y} + \dots. \\
\dot{\delta} &= g(x_* + \epsilon, y_* + \delta) \\
&= g + \epsilon\frac{\partial g}{\partial x} + \delta\frac{\partial g}{\partial y} + \dots \\
&= \epsilon\frac{\partial g}{\partial x} + \delta\frac{\partial g}{\partial y} + \dots,
\end{aligned}$$

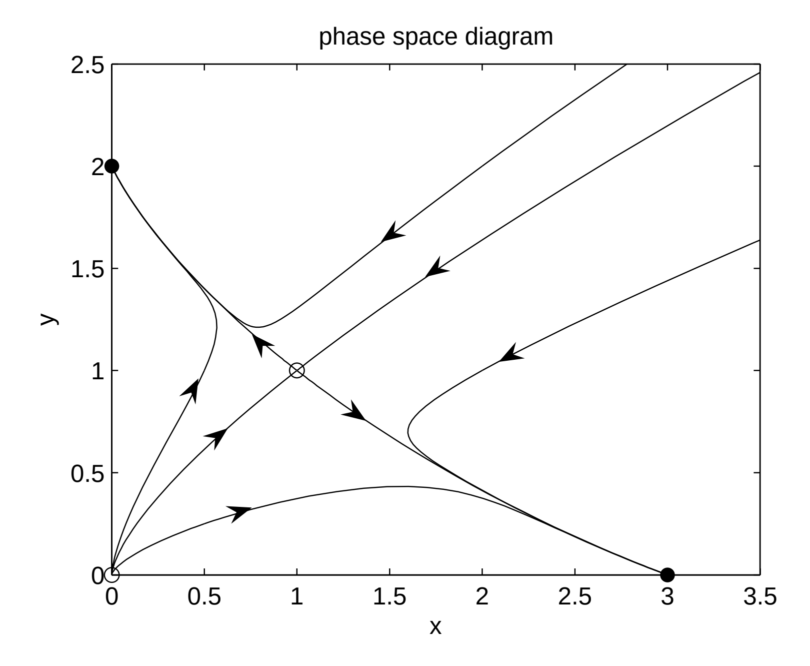


Figure 7.1: *Phase space plot for two-dimensional nonlinear system.*

where in the Taylor series f, g and all their partial derivatives are evaluated at the fixed point (x_*, y_*). Neglecting higher-order terms in the Taylor series, we thus have a system of odes for the perturbation, given in matrix form as

$$\frac{d}{dt}\begin{pmatrix} \epsilon \\ \delta \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} \epsilon \\ \delta \end{pmatrix}. \tag{7.3}$$

The two-by-two matrix in (7.3) is called the Jacobian matrix at the fixed point. An eigenvalue analysis of the Jacobian matrix will typically yield two eigenvalues λ_1 and λ_2. These eigenvalues may be real and distinct, complex conjugate pairs, or repeated. The fixed point is stable (all perturbations decay exponentially) if both eigenvalues have negative real parts. The fixed point is unstable (some perturbations grow exponentially) if at least one of the eigenvalues has a positive real part. Fixed points can be further classified as stable or unstable nodes, unstable saddle points, stable or unstable spiral points, or stable or unstable improper nodes.

Example: Find all the fixed points of the nonlinear system $\dot{x} = x(3 - x - 2y)$, $\dot{y} = y(2 - x - y)$, and determine their stability.

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The fixed points are determined by solving

$$f(x, y) = x(3 - x - 2y) = 0, \quad g(x, y) = y(2 - x - y) = 0.$$

Evidently, $(x, y) = (0, 0)$ is a fixed point. On the one hand, if only $x = 0$, then the equation $g(x, y) = 0$ yields $y = 2$. On the other hand, if only $y = 0$, then the equation $f(x, y) = 0$ yields $x = 3$. If both x and y are nonzero, then we must solve the linear system

$$x + 2y = 3, \quad x + y = 2,$$

and the solution is easily found to be $(x, y) = (1, 1)$. Hence, we have determined the four fixed points $(x_*, y_*) = (0, 0)$, $(0, 2)$, $(3, 0)$, $(1, 1)$. The Jacobian matrix is given by

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Stability of the fixed points may be considered in turn. With $\mathbf{J}_*$ the Jacobian matrix evaluated at the fixed point, we have

$$(x_*, y_*) = (0, 0) : \quad \mathbf{J}_* = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues of $\mathbf{J}_*$ are $\lambda = 3, 2$ so that the fixed point $(0, 0)$ is an unstable node. Next,

$$(x_*, y_*) = (0, 2) : \quad \mathbf{J}_* = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}.$$

The eigenvalues of $\mathbf{J}_*$ are $\lambda = -1, -2$ so that the fixed point $(0, 2)$ is a stable node. Next,

$$(x_*, y_*) = (3, 0) : \quad \mathbf{J}_* = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues of $\mathbf{J}_*$ are $\lambda = -3, -1$ so that the fixed point $(3, 0)$ is also a stable node. Finally,

$$(x_*, y_*) = (1, 1) : \quad \mathbf{J}_* = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}.$$

The characteristic equation of $\mathbf{J}_*$ is given by $(-1 - \lambda)^2 - 2 = 0$, so that $\lambda = -1 \pm \sqrt{2}$. Since one eigenvalue is negative and the other positive the fixed point $(1, 1)$ is an unstable saddle point. From our analysis of the fixed points, one can expect that all solutions will asymptote to one of the stable fixed points $(0, 2)$ or $(3, 0)$, depending on the initial conditions.

It is of interest to sketch the phase space diagram for this nonlinear system. The eigenvectors associated with the unstable saddle point $(1, 1)$ determine the directions of the flow into and away from this fixed point. The eigenvector associated with the positive eigenvalue $\lambda_1 = -1 + \sqrt{2}$ can be determined from the first equation of $(\mathbf{J}_* - \lambda_1 \mathbf{I})\mathbf{v_1} = 0$, or

$$-\sqrt{2} v_{11} - 2 v_{12} = 0,$$

so that $v_{12} = -(\sqrt{2}/2) v_{11}$. The eigenvector associated with the negative eigenvalue $\lambda_1 = -1 - \sqrt{2}$ satisfies $v_{22} = (\sqrt{2}/2) v_{21}$. The eigenvectors give the slope of the lines with origin at the fixed point for incoming (negative eigenvalue) and outgoing (positive eigenvalue) trajectories. The outgoing trajectories have negative slope $-\sqrt{2}/2$ and the incoming trajectories have positive slope $\sqrt{2}/2$. A rough sketch of the phase space diagram can be made by hand (as demonstrated in class). Here, a computer generated plot obtained from numerical solution of the nonlinear coupled odes is presented in Fig. 7.1. The curve starting from the origin and at infinity, and terminating at the unstable saddle point is called the separatrix. This curve separates the phase space into two regions: initial conditions for which the solution asymptotes to the fixed point $(0, 2)$, and initial conditions for which the solution asymptotes to the fixed point $(3, 0)$.

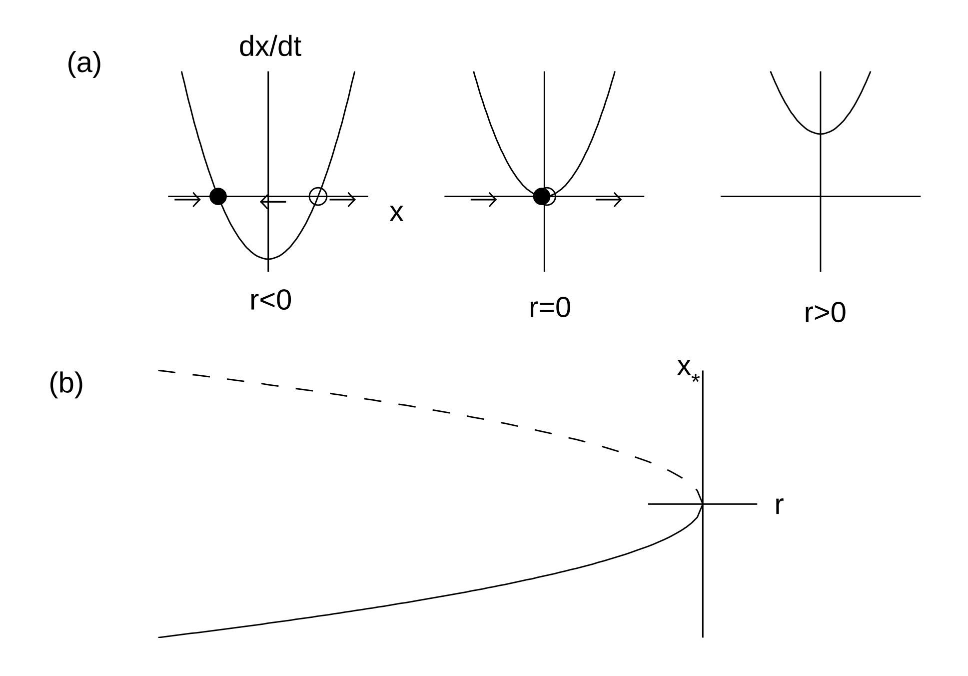


Figure 7.2: *Saddlenode bifurcation. (a) $\dot{x}$ versus x; (b) bifurcation diagram.*

7.2 One-dimensional bifurcations

A bifurcation occurs in a nonlinear differential equation when a small change in a parameter results in a qualitative change in the long-time solution. Examples of bifurcations are when fixed points are created or destroyed, or change their stability.

We now consider four classic bifurcations of one-dimensional nonlinear differential equations: saddle-node bifurcation, transcritical bifurcation, supercritical pitchfork bifurcation, and subcritical pitchfork bifurcation. The corresponding differential equation will be written as

$$\dot{x} = f_r(x),$$

where the subscript r represents a parameter that results in a bifurcation when varied across zero. The simplest differential equations that exhibit these bifurcations are called the *normal forms*, and correspond to a local analysis (i.e., Taylor series expansion) of more general differential equations around the fixed point, together with a possible rescaling of x.

7.2.1 Saddle-node bifurcation

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The saddle-node bifurcation results in fixed points being created or destroyed. The normal form for a saddle-node bifurcation is given by

$$\dot{x} = r + x^2.$$

The fixed points are $x_* = \pm\sqrt{-r}$. Clearly, two real fixed points exist when $r < 0$ and no real fixed points exist when $r > 0$. The stability of the fixed points when

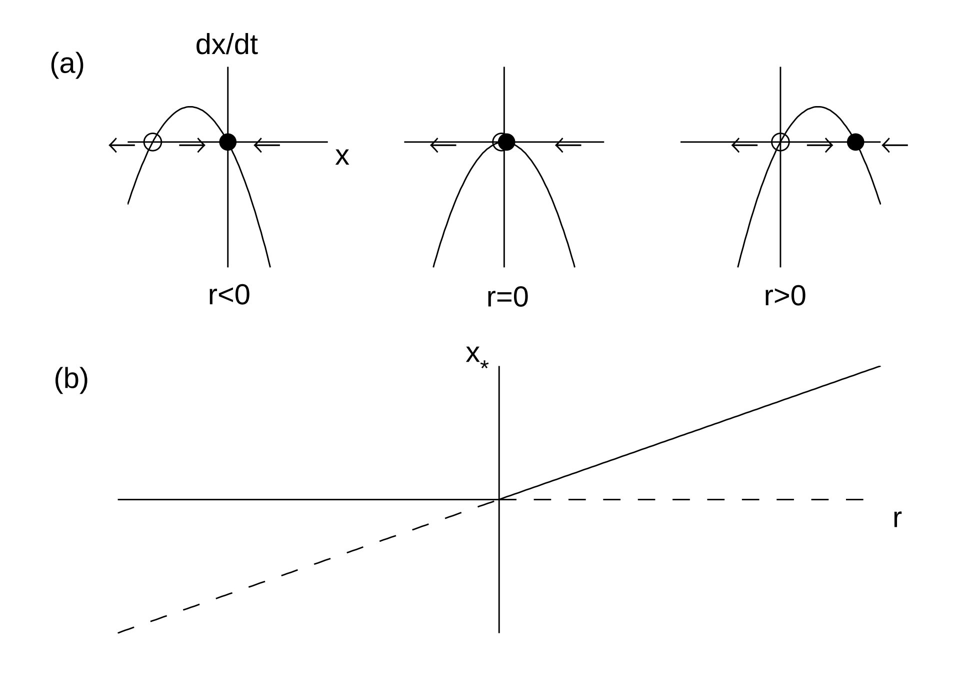


Figure 7.3: *Transcritical bifurcation. (a) $\dot{x}$ versus x; (b) bifurcation diagram.*

$r < 0$ are determined by the derivative of $f(x) = r + x^2$, given by $f'(x) = 2x$. Therefore, the negative fixed point is stable and the positive fixed point is unstable.

Graphically, we can illustrate this bifurcation in two ways. First, in Fig. 7.2(a), we plot $\dot{x}$ versus x for the three parameter values corresponding to $r < 0$, $r = 0$ and $r > 0$. The values at which $\dot{x} = 0$ correspond to the fixed points, and arrows are drawn indicating how the solution $x(t)$ evolves (to the right if $\dot{x} > 0$ and to the left if $\dot{x} < 0$). The stable fixed point is indicated by a filled circle and the unstable fixed point by an open circle. Note that when $r = 0$, solutions converge to the origin from the left, but diverge from the origin on the right. Second, in Fig. 7.2(b), we plot a bifurcation diagram illustrating the fixed point x_* versus the bifurcation parameter r. The stable fixed point is denoted by a solid line and the unstable fixed point by a dashed line. Note that the two fixed points collide and annihilate at $r = 0$, and there are no fixed points for $r > 0$.

7.2.2 Transcritical bifurcation

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A transcritical bifurcation occurs when there is an exchange of stabilities between two fixed points. The normal form for a transcritical bifurcation is given by

$$\dot{x} = rx - x^2.$$

The fixed points are $x_* = 0$ and $x_* = r$. The derivative of the right-hand-side is $f'(x) = r - 2x$, so that $f'(0) = r$ and $f'(r) = -r$. Therefore, for $r < 0$, $x_* = 0$ is stable and $x_* = r$ is unstable, while for $r > 0$, $x_* = r$ is stable and $x_* = 0$ is

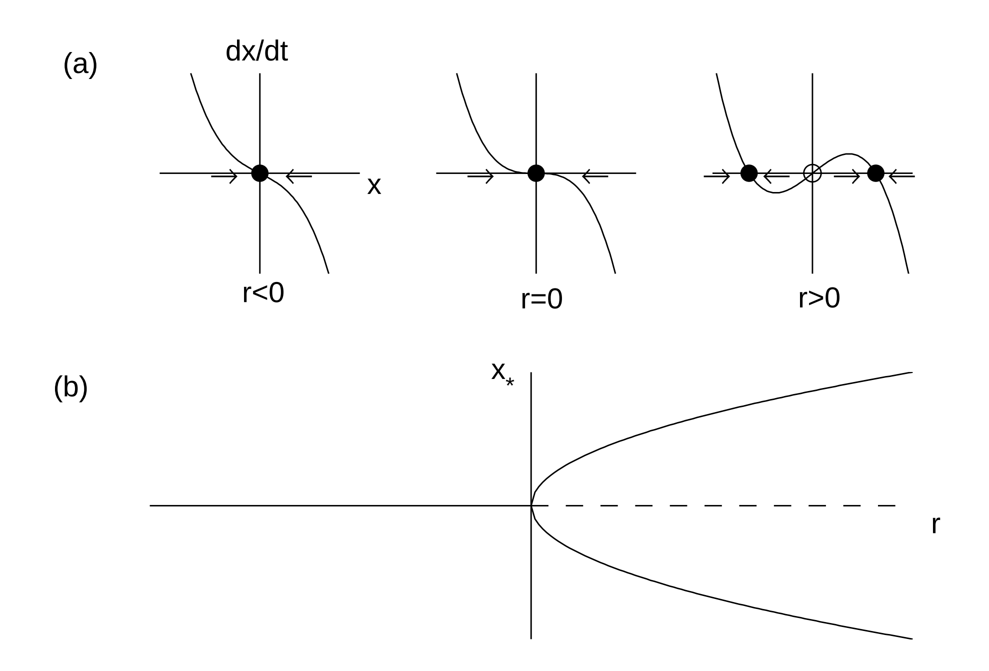


Figure 7.4: *Supercritical pitchfork bifurcation. (a) $\dot{x}$ versus x; (b) bifurcation diagram.*

unstable. The two fixed points thus exchange stability as r passes through zero. The transcritical bifurcation is illustrated in Fig. 7.3.

7.2.3 Supercritical pitchfork bifurcation

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The pitchfork bifurcations occur in physical models where fixed points appear and disappear in pairs due to some intrinsic symmetry of the problem. Pitchfork bifurcations can come in one of two types. In the supercritical bifurcation, a pair of stable fixed points are created at the bifurcation (or critical) point and exist after (super) the bifurcation. In the subcritical bifurcation, a pair of unstable fixed points are created at the bifurcation point and exist before (sub) the bifurcation.

The normal form for the supercritical pitchfork bifurcation is given by

$$\dot{x} = rx - x^3.$$

Note that the linear term results in exponential growth when $r > 0$ and the nonlinear term stabilizes this growth. The fixed points are $x_* = 0$ and $x_* = \pm\sqrt{r}$, the latter fixed points existing only when $r > 0$. The derivative of f is $f'(x) = r - 3x^2$ so that $f'(0) = r$ and $f'(\pm\sqrt{r}) = -2r$. Therefore, the fixed point $x_* = 0$ is stable for $r < 0$ and unstable for $r > 0$ while the fixed points $x = \pm\sqrt{r}$ exist and are stable for $r > 0$. Notice that the fixed point $x_* = 0$ becomes unstable as r crosses zero and two new stable fixed points $x_* = \pm\sqrt{r}$ are born. The supercritical pitchfork bifurcation is illustrated in Fig. 7.4.

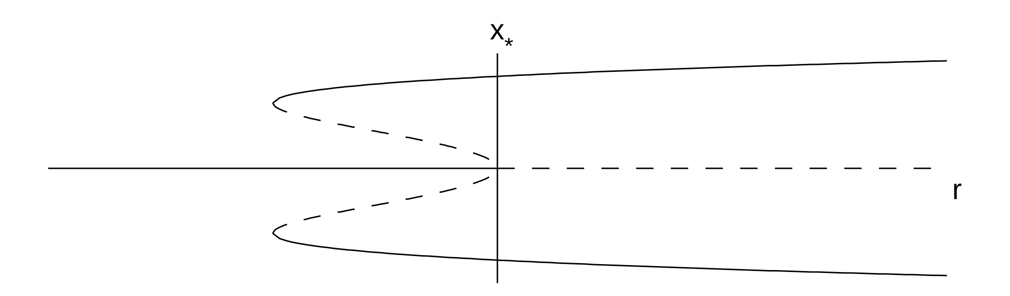


Figure 7.5: *Subcritical pitchfork bifurcation.*

7.2.4 Subcritical pitchfork bifurcation

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In the subcritical case, the cubic term is destabilizing. The normal form (to order x^3) is

$$\dot{x} = rx + x^3.$$

The fixed points are $x_* = 0$ and $x_* = \pm\sqrt{-r}$, the latter fixed points existing only when $r \leq 0$. The derivative of the right-hand-side is $f'(x) = r + 3x^2$ so that $f'(0) = r$ and $f'(\pm\sqrt{-r}) = -2r$. Therefore, the fixed point $x_* = 0$ is stable for $r < 0$ and unstable for $r > 0$ while the fixed points $x = \pm\sqrt{-r}$ exist and are unstable for $r < 0$. There are no stable fixed points when $r > 0$.

The absence of stable fixed points for $r > 0$ indicates that the neglect of terms of higher-order in x than x^3 in the normal form may be unwarranted. Keeping to the intrinsic symmetry of the equations (only odd powers of x) we can add a stabilizing nonlinear term proportional to x^5. The extended normal form (to order x^5) is

$$\dot{x} = rx + x^3 - x^5,$$

and is somewhat more difficult to analyze. The fixed points are solutions of

$$x(r + x^2 - x^4) = 0.$$

The fixed point $x_* = 0$ exists for all r, and four additional fixed points can be found from the solutions of the quadratic equation in x^2:

$$x_* = \pm\sqrt{\frac{1}{2}(1 \pm \sqrt{1+4r})}.$$

These fixed points exist only if x_* is real. Clearly, for the inner square-root to be real, $r \geq -1/4$. Also observe that $1 - \sqrt{1+4r}$ becomes negative for $r > 0$. We thus have three intervals in r to consider, and these regions and their fixed points are

$$r < -\frac{1}{4}: \quad x_* = 0 \quad \text{(one fixed point)};$$

$$-\frac{1}{4} < r < 0: \quad x_* = 0, \quad x_* = \pm\sqrt{\frac{1}{2}(1 \pm \sqrt{1+4r})} \quad \text{(five fixed points)};$$

$$r > 0: \quad x_* = 0, \quad x_* = \pm\sqrt{\frac{1}{2}(1 + \sqrt{1+4r})} \quad \text{(three fixed points)}.$$

Stability is determined from $f'(x) = r + 3x^2 - 5x^4$. Now, $f'(0) = r$ so $x_* = 0$ is stable for $r < 0$ and unstable for $r > 0$. The calculation for the other four roots can be simplified by noting that x_* satisfies $r + x_*^2 - x_*^4 = 0$, or $x_*^4 = r + x_*^2$. Therefore,

$$\begin{aligned} f'(x_*) &= r + 3x_*^2 - 5x_*^4 \\ &= r + 3x_*^2 - 5(r + x_*^2) \\ &= -4r - 2x_*^2 \\ &= -2(2r + x_*^2). \end{aligned}$$

With $x_*^2 = \frac{1}{2}(1 \pm \sqrt{1+4r})$, we have

$$\begin{aligned} f'(x_*) &= -2\left(2r + \frac{1}{2}(1 \pm \sqrt{1+4r})\right) \\ &= -\left((1+4r) \pm \sqrt{1+4r}\right) \\ &= -\sqrt{1+4r}\left(\sqrt{1+4r} \pm 1\right). \end{aligned}$$

Clearly, the plus root is always stable since $f'(x_*) < 0$. The minus root exists only for $-\frac{1}{4} < r < 0$ and is unstable since $f'(x_*) > 0$. We summarize the stability of the various fixed points:

$$\begin{aligned} r < -\frac{1}{4}: &\quad x_* = 0 \quad \text{(stable);} \\ -\frac{1}{4} < r < 0: &\quad x_* = 0, \quad \text{(stable)} \\ &\quad x_* = \pm\sqrt{\frac{1}{2}(1 + \sqrt{1+4r})} \quad \text{(stable);} \\ &\quad x_* = \pm\sqrt{\frac{1}{2}(1 - \sqrt{1+4r})} \quad \text{(unstable);} \\ r > 0: &\quad x_* = 0 \quad \text{(unstable)} \\ &\quad x_* = \pm\sqrt{\frac{1}{2}(1 + \sqrt{1+4r})} \quad \text{(stable).} \end{aligned}$$

The bifurcation diagram is shown in Fig. 7.5, and consists of a subcritical pitchfork bifurcation at $r = 0$ and two saddle-node bifurcations at $r = -1/4$. We can imagine what happens to x as r increases from negative values, supposing there is some small noise in the system so that $x = x(t)$ will diverge from unstable fixed points. For $r < -1/4$, the equilibrium value of x is $x_* = 0$. As r increases into the range $-1/4 < r < 0$, x will remain at $x_* = 0$. However, a catastrophe occurs as soon as $r > 0$. The $x_* = 0$ fixed point becomes unstable and the solution will jump up (or down) to the only remaining stable fixed point. Such behavior is called a jump bifurcation. A similar catastrophe can happen as r decreases from positive values. In this case, the jump occurs as soon as $r < -1/4$.

Since the stable equilibrium value of x depends on whether we are increasing or decreasing r, we say that the system exhibits *hysteresis*. The existence of a subcritical pitchfork bifurcation can be very dangerous in engineering applications since a small change in a problem's parameters can result in a large change in the equilibrium state. Physically, this can correspond to a collapse of a structure, or the failure of a component.

7.2.5 Application: a mathematical model of a fishery

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We illustrate the utility of bifurcation theory by analyzing a simple model of a fishery. We utilize the logistic equation (see §2.4.6) to model a fish population in the absence of fishing. To model fishing, we assume that the government has established fishing quotas so that at most a total of C fish per year may be caught. We assume that when the fish population is at the carrying capacity of the environment, fisherman can catch nearly their full quota. When the fish population drops to lower values, fish may be harder to find and the catch rate may fall below C, eventually going to zero as the fish population diminishes. Combining the logistic equation together with a simple model of fishing, we propose the mathematical model

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \frac{CN}{A+N}, \tag{7.4}$$

where N is the fish population size, t is time, r is the maximum potential growth rate of the fish population, K is the carrying capacity of the environment, C is the maximum rate at which fish can be caught, and A is a constant satisfying $A < K$ that is used to model the idea that fish become harder to catch when scarce.

We nondimensionalize (7.4) using $x = N/K$, $\tau = rt$, $c = C/rK$, $\alpha = A/K$:

$$\frac{dx}{d\tau} = x(1-x) - \frac{cx}{\alpha + x}. \tag{7.5}$$

Note that $0 \le x \le 1$, $c > 0$ and $0 < \alpha < 1$. We wish to qualitatively describe the equilibrium solutions of (7.5) and the bifurcations that may occur as the nondimensional catch rate c increases from zero. Practically, a government would like to issue each year as large a catch quota as possible without adversely affecting the number of fish that may be caught in subsequent years.

The fixed points of (7.5) are $x_* = 0$, valid for all c, and the solutions to $x^2 - (1 - \alpha)x + (c - \alpha) = 0$, or

$$x_* = \frac{1}{2}\left[(1-\alpha) \pm \sqrt{(1+\alpha)^2 - 4c}\right]. \tag{7.6}$$

The fixed points given by (7.6) are real only if $c < \frac{1}{4}(1+\alpha)^2$. Furthermore, the minus root is greater than zero only if $c > \alpha$. We therefore need to consider three intervals over which the following fixed points exist:

$$\begin{aligned}
0 \le c \le \alpha: &\quad x_* = 0, \quad x_* = \frac{1}{2}\left[(1-\alpha) + \sqrt{(1+\alpha)^2 - 4c}\right]; \\
\alpha < c < \frac{1}{4}(1+\alpha)^2: &\quad x_* = 0, \quad x_* = \frac{1}{2}\left[(1-\alpha) \pm \sqrt{(1+\alpha)^2 - 4c}\right]; \\
c > \frac{1}{4}(1+\alpha)^2: &\quad x_* = 0.
\end{aligned}$$

The stability of the fixed points can be determined with rigor analytically or graphically. Here, we simply apply biological intuition together with knowledge of the types of one dimensional bifurcations. An intuitive argument is made simpler if we consider c decreasing from large values. When c is large, that is $c > \frac{1}{4}(1+\alpha)^2$, too many fish are being caught and our intuition suggests that the fish population goes extinct. Therefore, in this interval, the single fixed point $x_* = 0$ must be stable. As

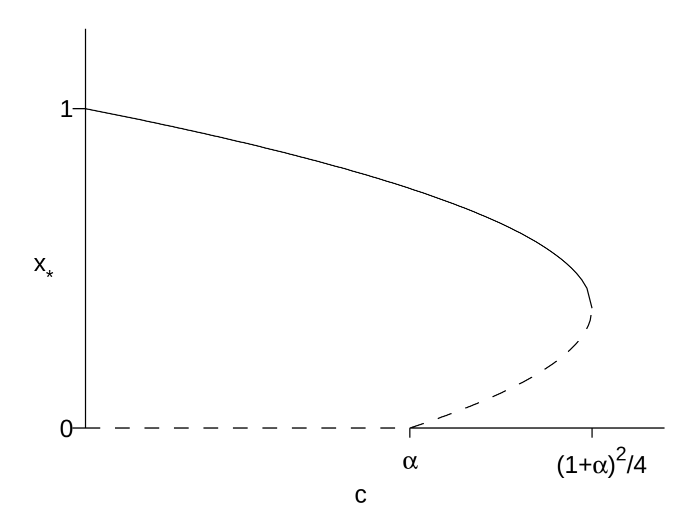


Figure 7.6: *Fishery bifurcation diagram.*

c decreases, a bifurcation occurs at $c = \frac{1}{4}(1+\alpha)^2$ introducing two additional fixed points at $x_* = (1-\alpha)/2$. The type of one dimensional bifurcation in which two fixed points are created as a square root becomes real is a saddlenode bifurcation, and one of the fixed points will be stable and the other unstable. Following these fixed points as $c \to 0$, we observe that the plus root goes to one, which is the appropriate stable fixed point when there is no fishing. We therefore conclude that the plus root is stable and the minus root is unstable. As c decreases further from this bifurcation, the minus root collides with the fixed point $x_* = 0$ at $c = \alpha$. This appears to be a transcritical bifurcation and assuming an exchange of stability occurs, we must have the fixed point $x_* = 0$ becoming unstable for $c < \alpha$. The resulting bifurcation diagram is shown in Fig. 7.6.

The purpose of simple mathematical models applied to complex ecological problems is to offer some insight. Here, we have learned that overfishing (in the model $c > \frac{1}{4}(1+\alpha)^2$) during one year can potentially result in a sudden collapse of the fish catch in subsequent years, so that governments need to be particularly cautious when contemplating increases in fishing quotas.

7.3 Two-dimensional bifurcations

All the one-dimensional bifurcations can also occur in two-dimensions along one of the directions. In addition, a new type of bifurcation can also occur in two-dimensions. Suppose there is some control parameter μ. Furthermore, suppose that for $\mu < 0$, a two-dimensional system approaches a fixed point by exponentially-damped oscillations. We know that the Jacobian matrix at the fixed point with $\mu < 0$ will have complex conjugate eigenvalues with negative real parts. Now suppose that when $\mu > 0$ the real parts of the eigenvalues become positive so that the fixed point becomes unstable. This change in stability of the fixed point is called

a *Hopf bifurcation*. The Hopf bifurcation comes in two types: supercritical Hopf bifurcation and subcritical Hopf bifurcation. For the supercritical Hopf bifurcation, as μ increases slightly above zero, the resulting oscillation around the now unstable fixed point is quickly stabilized at small amplitude. This stable orbit is called a *limit cycle*. For the subcritical Hopf bifurcation, as μ increases slightly above zero, the limit cycle immediately jumps to large amplitude.

7.3.1 Supercritical Hopf bifurcation

A simple example of a supercritical Hopf bifurcation can be given in polar coordinates:

$$\begin{aligned}\dot{r} &= \mu r - r^3,\\ \dot{\theta} &= \omega + br^2,\end{aligned}$$

where $x = r\cos\theta$ and $y = r\sin\theta$. The parameter μ controls the stability of the fixed point at the origin, the parameter ω is the frequency of oscillation near the origin, and the parameter b determines the dependence of the oscillation frequency at larger amplitude oscillations. Although we include b for generality, our qualitative analysis of these equations will be independent of b.

The equation for the radius is of the form of the supercritical pitchfork bifurcation. The fixed points are $r_* = 0$ and $r_* = \sqrt{\mu}$ (note that $r > 0$), and the former fixed point is stable for $\mu < 0$ and the latter is stable for $\mu > 0$. The transition of the eigenvalues of the Jacobian from negative real part to positive real part can be seen if we transform these equations to cartesian coordinates. We have using $r^2 = x^2 + y^2$,

$$\begin{aligned}\dot{x} &= \dot{r}\cos\theta - \dot{\theta}r\sin\theta\\ &= (\mu r - r^3)\cos\theta - (\omega + br^2)r\sin\theta\\ &= \mu x - (x^2+y^2)x - \omega y - b(x^2+y^2)y\\ &= \mu x - \omega y - (x^2+y^2)(x+by);\\ \dot{y} &= \dot{r}\sin\theta + \dot{\theta}r\cos\theta\\ &= (\mu r - r^3)\sin\theta + (\omega + br^2)r\cos\theta\\ &= \mu y - (x^2+y^2)y + \omega x + b(x^2+y^2)x\\ &= \omega x + \mu y - (x^2+y^2)(y - bx).\end{aligned}$$

The stability of the origin is determined by the Jacobian matrix evaluated at the origin. The nonlinear terms in the equation vanish and the Jacobian matrix at the origin is given by

$$J = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}.$$

The eigenvalues are the solutions of $(\mu - \lambda)^2 + \omega^2 = 0$, or $\lambda = \mu \pm i\omega$. As μ increases from negative to positive values, exponentially damped oscillations change into exponentially growing oscillations. The nonlinear terms in the equations stabilize the growing oscillations into a limit cycle.

7.3.2 Subcritical Hopf bifurcation

The analogous example of a subcritical Hopf bifurcation is given by

$$\dot{r} = \mu r + r^3 - r^5,$$
$$\dot{\theta} = \omega + br^2.$$

Here, the equation for the radius is of the form of the subcritical pitchfork bifurcation. As μ increases from negative to positive values, the origin becomes unstable and exponentially growing oscillations increase until the radius reaches a stable fixed point far away from the origin. In practice, it may be difficult to tell analytically if a Hopf bifurcation is supercritical or subcritical from the equations of motion. Computational solution, however, can quickly distinguish between the two types.

Chapter 8

Partial differential equations

Reference: Boyce and DiPrima, Chapter 10

Differential equations containing partial derivatives with two or more independent variables are called partial differential equations (pdes). These equations are of fundamental scientific interest but are substantially more difficult to solve, both analytically and computationally, than odes.

In this chapter, we begin by deriving two fundamental pdes: the diffusion equation and the wave equation, and show how to solve them with prescribed boundary conditions using the technique of separation of variables. We then discuss solutions of the two-dimensional Laplace equation in cartesian and polar coordinates, and finish with a lengthy discussion of the Schrödinger equation, a partial differential equation fundamental to both physics and chemistry.

8.1 Derivation of the diffusion equation

To derive the diffusion equation in one spacial dimension, we imagine a still liquid in a long pipe of constant cross sectional area. A small quantity of dye is placed in a cross section of the pipe and allowed to diffuse up and down the pipe. The dye diffuses from regions of higher concentration to regions of lower concentration.

We define $u(x,t)$ to be the concentration of the dye at position x along the pipe, and we wish to find the pde satisfied by u. It is useful to keep track of the units of the various quantities involved in the derivation and we introduce the bracket notation $[X]$ to mean the units of X. Relevant dimensional units used in the derivation of the diffusion equation are mass m, length l, and time t. Assuming that the dye concentration is uniform in every cross section of the pipe, the dimensions of concentration used here are $[u] = m/l$.

The mass of dye in the infinitesimal pipe volume located between position x_1 and position x_2 at time t, with $x_1 < x < x_2$, is given to order $\Delta x = x_2 - x_1$ by

$$M = u(x,t)\Delta x.$$

The mass of dye in this infinitesimal pipe volume changes by diffusion into and out of the cross sectional ends situated at position x_1 and x_2 (Figure 8.1). We assume the rate of diffusion is proportional to the concentration gradient, a relationship known as Fick's law of diffusion. Fick's law of diffusion assumes the mass flux J, with units $[J] = m/t$ across a cross section of the pipe is given by

$$J = -Du_x, \tag{8.1}$$

where the diffusion constant $D > 0$ has units $[D] = l^2/t$, and we have used the notation $u_x = \partial u/\partial x$. The mass flux is opposite in sign to the gradient of concentration so that the flux is from high concentration to low concentration. The time rate of change in the mass of dye between x_1 and x_2 is given by the difference between the mass flux into and the mass flux out of the infinitesimal cross sectional

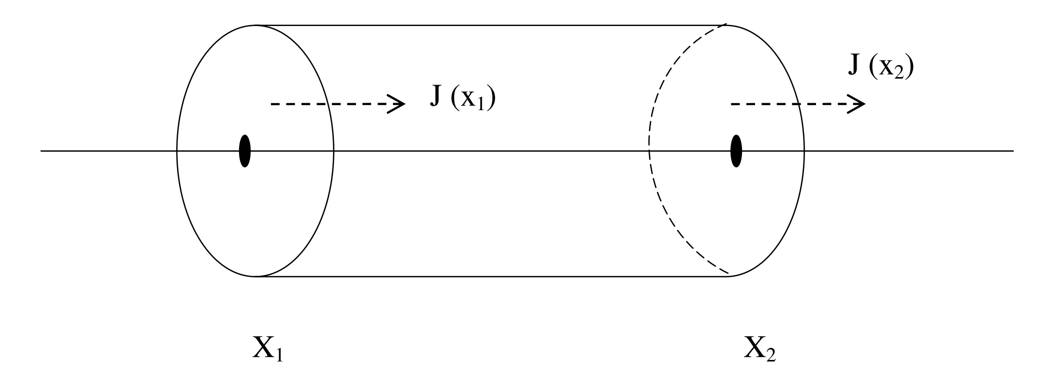


Figure 8.1: *Derivation of the diffusion equation.*

volume. A positive mass flux signifies diffusion from left to right. Therefore, the time rate of change of the dye mass is given by

$$\frac{dM}{dt} = J(x_1,t) - J(x_2,t),$$

or rewriting in terms of $u(x,t)$:

$$u_t(x,t)\Delta x = D\left(u_x(x_2,t) - u_x(x_1,t)\right).$$

Dividing by Δx and taking the limit $\Delta x \to 0$ results in the diffusion equation:

$$u_t = Du_{xx}.$$

We note that the diffusion equation is identical to the heat conduction equation, where u is temperature, and the constant D (commonly written as κ) is the thermal conductivity.

8.2 Derivation of the wave equation

To derive the wave equation in one spacial dimension, we imagine an elastic string that undergoes small amplitude transverse vibrations. We define $u(x,t)$ to be the vertical displacement of the string from the x-axis at position x and time t, and we wish to find the pde satisfied by u. We define ρ to be the constant mass density of the string, T the tension of the string, and θ the angle between the string and the horizonal line. We consider an infinitesimal string element located between x_1 and x_2, with $\Delta x = x_2 - x_1$, as shown in Fig. 8.2. The governing equations are Newton's law of motion for the horizontal and vertical acceleration of our infinitesimal string element, and we assume that the string element only accelerates vertically. Therefore, the horizontal forces must balance and we have

$$T_2\cos\theta_2 = T_1\cos\theta_1.$$

The vertical forces result in a vertical acceleration, and with u_{tt} the vertical acceleration of the string element and $\rho\sqrt{\Delta x^2 + \Delta u^2} = \rho\Delta x\sqrt{1+u_x^2}$ its mass, where we have used $u_x = \Delta u/\Delta x$, exact as $\Delta x \to 0$, we have

$$\rho\Delta x\sqrt{1+u_x^2}\,u_{tt} = T_2\sin\theta_2 - T_1\sin\theta_1.$$

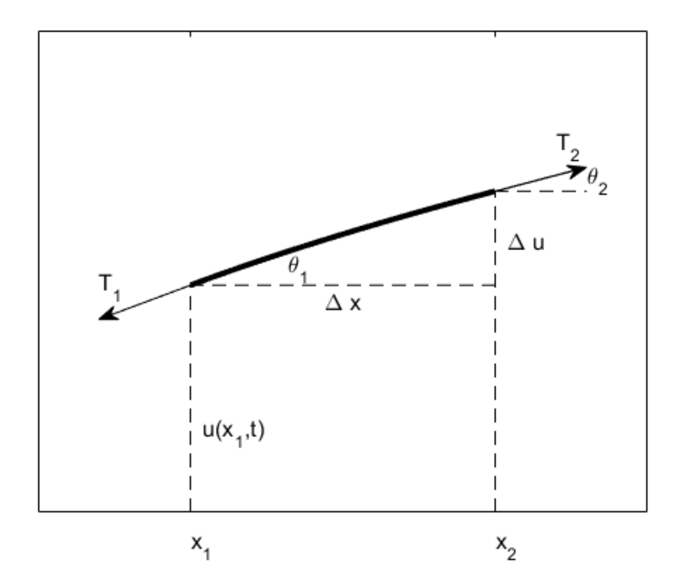


Figure 8.2: *Derivation of the wave equation.*

We now make the assumption of small vibrations, that is $\Delta u \ll \Delta x$, or equivalently $u_x \ll 1$. Note that $[u] = l$ so that u_x is dimensionless. With this approximation, to leading-order in u_x we have

$$\cos\theta_2 = \cos\theta_1 = 1,$$

$$\sin\theta_2 = u_x(x_2, t), \quad \sin\theta_1 = u_x(x_1, t),$$

and

$$\sqrt{1 + u_x^2} = 1.$$

Therefore, to leading order $T_1 = T_2 = T$, (i.e., the tension in the string is approximately constant), and

$$\rho \Delta x u_{tt} = T\left(u_x(x_2, t) - u_x(x_1, t)\right).$$

Dividing by Δx and taking the limit $\Delta x \to 0$ results in the wave equation

$$u_{tt} = c^2 u_{xx},$$

where $c^2 = T/\rho$. Since $[T] = ml/t^2$ and $[\rho] = m/l$, we have $[c^2] = l^2/t^2$ so that c has units of velocity.

8.3 Fourier series

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Our solution of the diffusion and wave equations will require use of a Fourier series. A periodic function $f(x)$ with period $2L$, can be represented as a Fourier series in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right). \tag{8.2}$$

Determination of the coefficients $a_0, a_1, a_2, \ldots$ and $b_1, b_2, b_3, \ldots$ makes use of orthogonality relations for sine and cosine. We first define the widely used Kronecker delta δ_{nm} as

$$\delta_{nm} = \begin{cases} 1 & \text{if } n = m; \\ 0 & \text{otherwise.} \end{cases}$$

The orthogonality relations for n and m positive integers are then given with compact notation as the integration formulas

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = L\delta_{nm}, \tag{8.3}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = L\delta_{nm}, \tag{8.4}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 0. \tag{8.5}$$

To illustrate the integration technique used to obtain these results, we derive (8.4) assuming that n and m are positive integers with $n \neq m$. Changing variables to $\xi = \pi x/L$, we obtain

$$\begin{aligned}
&\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx \\
&\quad = \frac{L}{\pi} \int_{-\pi}^{\pi} \sin(m\xi) \sin(n\xi) d\xi \\
&\quad = \frac{L}{2\pi} \int_{-\pi}^{\pi} \left[\cos\left((m-n)\xi\right) - \cos\left((m+n)\xi\right)\right] d\xi \\
&\quad = \frac{L}{2\pi} \left[\frac{1}{m-n} \sin\left((m-n)\xi\right) - \frac{1}{m+n} \sin\left((m+n)\xi\right)\right]_{-\pi}^{\pi} \\
&\quad = 0.
\end{aligned}$$

For $m = n$, however,

$$\begin{aligned}
\int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx &= \frac{L}{\pi} \int_{-\pi}^{\pi} \sin^2(n\xi) d\xi \\
&= \frac{L}{2\pi} \int_{-\pi}^{\pi} \left(1 - \cos(2n\xi)\right) d\xi \\
&= \frac{L}{2\pi} \left[\xi - \frac{1}{2n} \sin 2n\xi\right]_{-\pi}^{\pi} \\
&= L.
\end{aligned}$$

Integration formulas given by (8.3) and (8.5) can be similarly derived.

To determine the coefficient a_n, we multiply both sides of (8.2) by $\cos(n\pi x/L)$ with n a nonnegative integer, and change the dummy summation variable from n to m. Integrating over x from $-L$ to L and assuming that the integration can be done term by term in the infinite sum, we obtain

$$\begin{aligned}
\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L} dx &= \frac{a_0}{2} \int_{-L}^{L} \cos\frac{n\pi x}{L} dx \\
&\quad + \sum_{m=1}^{\infty} \left\{ a_m \int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} dx + b_m \int_{-L}^{L} \cos\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx \right\}.
\end{aligned}$$

If $n = 0$, then the second and third integrals on the right-hand-side are zero and the first integral is $2L$ so that the right-hand-side becomes La_0. If n is a positive integer, then the first and third integrals on the right-hand-side are zero, and the second integral is $L\delta_{nm}$. For this case, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx &= \sum_{m=1}^{\infty} La_m\delta_{nm} \\ &= La_n,\end{aligned}$$

where all the terms in the summation except $m = n$ are zero by virtue of the Kronecker delta. We therefore obtain for $n = 0, 1, 2, \ldots$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx. \tag{8.6}$$

To determine the coefficients $b_1, b_2, b_3, \ldots$, we multiply both sides of (8.2) by $\sin(n\pi x/L)$, with n a positive integer, and again change the dummy summation variable from n to m. Integrating, we obtain

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx &= \frac{a_0}{2}\int_{-L}^{L}\sin\frac{n\pi x}{L}dx \\ &+ \sum_{m=1}^{\infty}\left\{a_m\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx + b_m\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx\right\}.\end{aligned}$$

Here, the first and second integrals on the right-hand-side are zero, and the third integral is $L\delta_{nm}$ so that

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx &= \sum_{m=1}^{\infty} Lb_m\delta_{nm} \\ &= Lb_n.\end{aligned}$$

Hence, for $n = 1, 2, 3, \ldots$,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \tag{8.7}$$

Our results for the Fourier series of a function $f(x)$ with period $2L$ are thus given by (8.2), (8.6) and (8.7).

8.4 Fourier cosine and sine series

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The Fourier series simplifies if $f(x)$ is an even function such that $f(-x) = f(x)$, or an odd function such that $f(-x) = -f(x)$. Use will be made of the following facts. The function $\cos(n\pi x/L)$ is an even function and $\sin(n\pi x/L)$ is an odd function. The product of two even functions is an even function. The product of two odd functions is an even function. The product of an even and an odd function is an odd function. And if $g(x)$ is an even function, then

$$\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx;$$

and if $g(x)$ is an odd function, then

$$\int_{-L}^{L} g(x)dx = 0.$$

We examine in turn the Fourier series for an even or an odd function. First, if $f(x)$ is even, then from (8.6) and (8.7) and our facts about even and odd functions,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \\ b_n &= 0. \end{aligned} \tag{8.8}$$

The Fourier series for an even function with period $2L$ is thus given by the Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad f(x) \text{ even.} \tag{8.9}$$

Second, if $f(x)$ is odd, then

$$\begin{aligned} a_n &= 0, \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx; \end{aligned} \tag{8.10}$$

and the Fourier series for an odd function with period $2L$ is given by the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad f(x) \text{ odd.} \tag{8.11}$$

Examples of Fourier series computed numerically can be obtained using the Java applet found at http://www.falstad.com/fourier. Here, we demonstrate an analytical example.

Example: Determine the Fourier cosine series of the even triangle function represented by Fig. 8.3.

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The triangle function depicted in Fig. 8.3 is an even function of x with period 2π (i.e., $L = \pi$). Its definition on $0 < x < \pi$ is given by

$$f(x) = 1 - \frac{2x}{\pi}.$$

Because $f(x)$ is even, it can be represented by the Fourier cosine series given by (8.8) and (8.9). The coefficient a_0 is

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} f(x)dx \\ &= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx \\ &= \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi} \\ &= 0. \end{aligned}$$

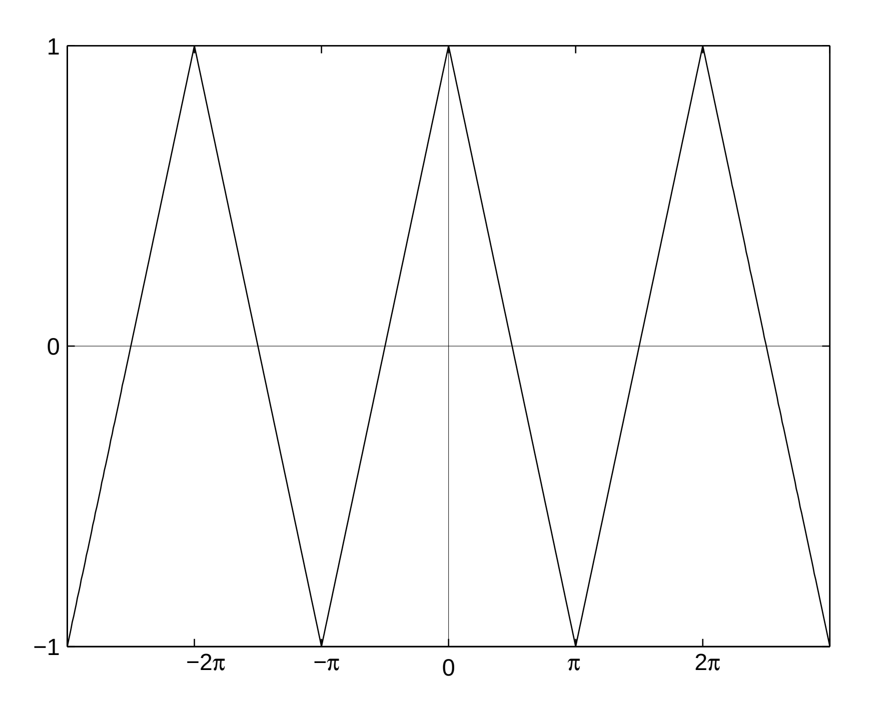


Figure 8.3: *The even triangle function.*

Note that $a_0/2$ is the average value of $f(x)$ over one period, and it is obvious from Fig. 8.3 that this value is zero. The coefficients for $n > 0$ are

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx \\
&= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos(nx)dx \\
&= \frac{2}{\pi}\int_0^{\pi}\cos(nx)dx - \frac{4}{\pi^2}\int_0^{\pi} x\cos(nx)dx \\
&= \frac{2}{n\pi}\sin(nx)\Big]_0^{\pi} - \frac{4}{\pi^2}\left\{\left[\frac{x}{n}\sin(nx)\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)dx\right\} \\
&= \frac{4}{n\pi^2}\int_0^{\pi}\sin(nx)dx \\
&= -\frac{4}{n^2\pi^2}\cos(nx)\Big]_0^{\pi} \\
&= \frac{4}{n^2\pi^2}\big(1-\cos(n\pi)\big).
\end{aligned}$$

Since

$$\cos(n\pi) = \begin{cases} -1, & \text{if } n \text{ odd;} \\ 1, & \text{if } n \text{ even;} \end{cases}$$

we have

$$a_n = \begin{cases} 8/(n^2\pi^2), & \text{if } n \text{ odd;} \\ 0, & \text{if } n \text{ even.} \end{cases}$$

The Fourier cosine series for the triangle function is therefore given by

$$f(x) = \frac{8}{\pi^2}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right).$$

Convergence of this series is rapid. As an interesting aside, evaluation of this series at $x = 0$, using $f(0) = 1$, yields an infinite series for $\pi^2/8$:

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots.$$

With Fourier series now included in our applied mathematics toolbox, we are ready to solve the diffusion and wave equations in bounded domains.

8.5 Solution of the diffusion equation

8.5.1 Homogeneous boundary conditions

We consider one dimensional diffusion in a pipe of length L, and solve the diffusion equation for the concentration $u(x,t)$,

$$u_t = Du_{xx}, \quad 0 \le x \le L, \quad t > 0. \tag{8.12}$$

Both initial and boundary conditions are required for a unique solution. That is, we assume the initial concentration distribution in the pipe is given by

$$u(x,0) = f(x), \quad 0 \le x \le L. \tag{8.13}$$

Furthermore, we assume that boundary conditions are given at the ends of the pipes. When the concentration value is specified at the boundaries, the boundary conditions are called *Dirichlet boundary conditions*. As the simplest example, we assume here homogeneous Dirichlet boundary conditions, that is zero concentration of dye at the ends of the pipe, which could occur if the ends of the pipe open up into large reservoirs of clear solution,

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0. \tag{8.14}$$

We will later also discuss inhomogeneous Dirichlet boundary conditions and homogeneous *Neumann boundary conditions*, for which the derivative of the concentration is specified to be zero at the boundaries. Note that if $f(x)$ is identically zero, then the trivial solution $u(x,t) = 0$ satisfies the differential equation and the initial and boundary conditions and is therefore the unique solution of the problem. In what follows, we will assume that $f(x)$ is not identically zero so that we need to find a solution different than the trivial solution.

The solution method we use is called *separation of variables*. We assume that $u(x,t)$ can be written as a product of two other functions, one dependent only on position x and the other dependent only on time t. That is, we make the ansatz

$$u(x,t) = X(x)T(t). \tag{8.15}$$

Whether this ansatz will succeed depends on whether the solution indeed has this form. Substituting (8.15) into (8.12), we obtain

$$XT' = DX''T,$$

which we rewrite by separating the x and t dependence to opposite sides of the equation:

$$\frac{X''}{X} = \frac{1}{D}\frac{T'}{T}.$$

The left hand side of this equation is independent of t and the right hand side is independent of x. Both sides of this equation are therefore independent of both x and t and equal to a constant. Introducing $-\lambda$ as the separation constant, we have

$$\frac{X''}{X} = \frac{1}{D}\frac{T'}{T} = -\lambda,$$

and we obtain the two ordinary differential equations

$$X'' + \lambda X = 0, \quad T' + \lambda DT = 0. \tag{8.16}$$

Because of the boundary conditions, we must first consider the equation for $X(x)$. To solve, we need to determine the boundary conditions at $x = 0$ and $x = L$. Now, from (8.14) and (8.15),

$$u(0,t) = X(0)T(t) = 0, \quad t > 0.$$

Since $T(t)$ is not identically zero for all t (which would result in the trivial solution for u), we must have $X(0) = 0$. Similarly, the boundary condition at $x = L$ requires $X(L) = 0$. We therefore consider the two-point boundary value problem

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0. \tag{8.17}$$

The equation given by (8.17) is called an ode eigenvalue problem. Clearly, the trivial solution $X(x) = 0$ is a solution. Nontrivial solutions exist only for discrete values of λ. These discrete values of λ and the corresponding functions $X(x)$ are called the eigenvalues and eigenfunctions of the differential equation.

Since the form of the general solution of the ode depends on the sign of λ, we consider in turn the cases $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$. For $\lambda > 0$, we write $\lambda = \mu^2$ and determine the general solution of

$$X'' + \mu^2 X = 0$$

to be

$$X(x) = A\cos\mu x + B\sin\mu x.$$

Applying the boundary condition at $x = 0$, we find $A = 0$. The boundary condition at $x = L$ then yields

$$B\sin\mu L = 0.$$

The solution $B = 0$ results in the trivial solution for u and can be ruled out. Therefore, we must have

$$\sin\mu L = 0,$$

which is an equation for μ. The solutions are

$$\mu = n\pi/L,$$

where n is an integer. We have thus determined the eigenvalues $\lambda = \mu^2 > 0$ to be

$$\lambda_n = (n\pi/L)^2, \quad n = 1, 2, 3, \ldots, \tag{8.18}$$

with corresponding eigenfunctions

$$X_n = \sin{(n\pi x/L)}. \tag{8.19}$$

For $\lambda < 0$, we write $\lambda = -\mu^2$ and determine the general solution of

$$X'' - \mu^2 X = 0$$

to be

$$X(x) = A\cosh\mu x + B\sinh\mu x,$$

where we have previously introduced the hyperbolic sine and cosine functions in §3.4.1. Applying the boundary condition at $x = 0$, we find $A = 0$. The boundary condition at $x = L$ then yields

$$B\sinh\mu L = 0,$$

which for $\mu \neq 0$ has only the solution $B = 0$. Therefore, there is no nontrivial solution for u with $\lambda < 0$. Finally, for $\lambda = 0$, we have

$$X'' = 0,$$

with general solution

$$X(x) = A + Bx.$$

The boundary condition at $x = 0$ and $x = L$ yields $A = B = 0$ so again there is no nontrivial solution for u with $\lambda = 0$.

We now turn to the equation for $T(t)$. The equation corresponding to the eigenvalue λ_n, using (8.18), is given by

$$T' + \left(n^2\pi^2 D/L^2\right) T = 0,$$

which has solution proportional to

$$T_n = e^{-n^2\pi^2 Dt/L^2}. \tag{8.20}$$

Therefore, multiplying the solutions given by (8.19) and (8.20), we conclude that the functions

$$u_n(x,t) = \sin{(n\pi x/L)}e^{-n^2\pi^2 Dt/L^2} \tag{8.21}$$

satisfy the pde given by (8.12) and the boundary conditions given by (8.14) for every positive integer n.

The principle of linear superposition for homogeneous linear differential equations then states that the general solution to (8.12) and (8.14) is given by

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n u_n(x,t) \\ &= \sum_{n=1}^{\infty} b_n \sin{(n\pi x/L)}e^{-n^2\pi^2 Dt/L^2}. \end{aligned} \tag{8.22}$$

The final solution step is to satisfy the initial conditions given by (8.13). At $t = 0$, we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin{(n\pi x/L)}. \tag{8.23}$$

We immediately recognize (8.23) as a Fourier sine series (8.11) for an odd function $f(x)$ with period $2L$. Equation (8.23) is a periodic extension of our original $f(x)$ defined on $0 \le x \le L$, and is an odd function because of the boundary condition $f(0) = 0$. From our solution for the coefficients of a Fourier sine series (8.10), we determine

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \tag{8.24}$$

Thus the solution to the diffusion equation with homogeneous Dirichlet boundary conditions defined by (8.12), (8.13) and (8.14) is given by (8.22) with the b_n coefficients computed from (8.24).

Example: Determine the concentration of a dye in a pipe of length L, where the dye has total initial mass M_0 and is initially concentrated at the center of the pipe, and the ends of the pipe are held at zero concentration.

The governing equation for concentration is the diffusion equation. We model the initial concentration of the dye by a delta-function centered at $x = L/2$, that is, $u(x,0) = f(x) = M_0\delta(x - L/2)$. Therefore, from (8.24),

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L M_0\delta(x - \frac{L}{2})\sin\frac{n\pi x}{L}dx \\ &= \frac{2M_0}{L}\sin(n\pi/2) \\ &= \begin{cases} 2M_0/L & \text{if } n = 1, 5, 9, \ldots; \\ -2M_0/L & \text{if } n = 3, 7, 11, \ldots; \\ 0 & \text{if } n = 2, 4, 6, \ldots. \end{cases} \end{aligned}$$

With b_n determined, the solution for $u(x,t)$ given by (8.22) can be written as

$$u(x,t) = \frac{2M_0}{L}\sum_{n=0}^{\infty}(-1)^n \sin\left(\frac{(2n+1)\pi x}{L}\right)e^{-(2n+1)^2\pi^2 Dt/L^2}.$$

When $t \gg L^2/D$, the leading-order term in the series is a good approximation and is given by

$$u(x,t) \approx \frac{2M_0}{L}\sin(\pi x/L)e^{-\pi^2 Dt/L^2}.$$

The mass of the dye in the pipe is decreasing in time, diffusing into the reservoirs located at both ends. The total mass in the pipe at time t can be found from

$$M(t) = \int_0^L u(x,t)dx,$$

and since

$$\int_0^L \sin(\pi x/L)dx = \frac{2L}{\pi},$$

we have for large times

$$M(t) = \frac{4M_0}{\pi}e^{-\pi^2 Dt/L^2}.$$

8.5.2 Inhomogeneous boundary conditions

Consider a diffusion problem where one end of the pipe has dye of concentration held constant at C_1 and the other held constant at C_2, which could occur if the ends of the pipe had large reservoirs of fluid with different concentrations of dye. With $u(x,t)$ the concentration of dye, the boundary conditions are given by

$$u(0,t) = C_1, \quad u(L,t) = C_2, \quad t > 0.$$

The concentration $u(x,t)$ satisfies the diffusion equation with diffusivity D:

$$u_t = Du_{xx}.$$

If we try to solve this problem directly using separation of variables, we will run into trouble. Applying the inhomogeneous boundary condition at $x = 0$ directly to the ansatz $u(x,t) = X(x)T(t)$ results in

$$u(0,t) = X(0)T(t) = C_1;$$

so that

$$X(0) = C_1/T(t).$$

However, our separation of variables ansatz assumes $X(x)$ to be independent of t! We therefore say that inhomogeneous boundary conditions are not separable.

The proper way to solve a problem with inhomogeneous boundary conditions is to transform it into another problem with homogeneous boundary conditions. As $t \to \infty$, we assume that a stationary concentration distribution $v(x)$ will attain, independent of t. Since $v(x)$ must satisfy the diffusion equation, we have

$$v''(x) = 0, \quad 0 \leq x \leq L,$$

with general solution

$$v(x) = A + Bx.$$

Since $v(x)$ must satisfy the same boundary conditions of $u(x,t)$, we have $v(0) = C_1$ and $v(L) = C_2$, and we determine $A = C_1$ and $B = (C_2 - C_1)/L$.

We now express $u(x,t)$ as the sum of the known asymptotic stationary concentration distribution $v(x)$ and an unknown transient concentration distribution $w(x,t)$:

$$u(x,t) = v(x) + w(x,t).$$

Substituting into the diffusion equation, we obtain

$$\frac{\partial}{\partial t}\left(v(x) + w(x,t)\right) = D\frac{\partial^2}{\partial x^2}\left(v(x) + w(x,t)\right)$$

or

$$w_t = Dw_{xx},$$

since $v_t = 0$ and $v_{xx} = 0$. The boundary conditions satisfied by w are

$$\begin{aligned} w(0,t) &= u(0,t) - v(0) = 0, \\ w(L,t) &= u(L,t) - v(L) = 0, \end{aligned}$$

so that w is observed to satisfy homogeneous boundary conditions. If the initial conditions are given by $u(x,0) = f(x)$, then the initial conditions for w are

$$\begin{aligned} w(x,0) &= u(x,0) - v(x) \\ &= f(x) - v(x). \end{aligned}$$

The resulting equations may then be solved for $w(x,t)$ using the technique for homogeneous boundary conditions, and $u(x,t)$ subsequently determined.

8.5.3 Pipe with closed ends

There is no diffusion of dye through the ends of a sealed pipe. Accordingly, the mass flux of dye through the pipe ends, given by (8.1), is zero so that the boundary conditions on the dye concentration $u(x,t)$ becomes

$$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0, \tag{8.25}$$

which are known as homogeneous Neumann boundary conditions. Again, we apply the method of separation of variables and as before, we obtain the two ordinary differential equations given by (8.16). Considering first the equation for $X(x)$, the appropriate boundary conditions are now on the first derivative of $X(x)$, and we must solve

$$X'' + \lambda X = 0, \quad X'(0) = X'(L) = 0. \tag{8.26}$$

Again, we consider in turn the cases $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$. For $\lambda > 0$, we write $\lambda = \mu^2$ and determine the general solution of (8.26) to be

$$X(x) = A\cos\mu x + B\sin\mu x,$$

so that taking the derivative

$$X'(x) = -\mu A\sin\mu x + \mu B\cos\mu x.$$

Applying the boundary condition $X'(0) = 0$, we find $B = 0$. The boundary condition at $x = L$ then yields

$$-\mu A\sin\mu L = 0.$$

The solution $A = 0$ results in the trivial solution for u and can be ruled out. Therefore, we must have

$$\sin\mu L = 0,$$

with solutions

$$\mu = n\pi/L,$$

where n is an integer. We have thus determined the eigenvalues $\lambda = \mu^2 > 0$ to be

$$\lambda_n = (n\pi/L)^2, \quad n = 1, 2, 3, \ldots, \tag{8.27}$$

with corresponding eigenfunctions

$$X_n = \cos(n\pi x/L). \tag{8.28}$$

For $\lambda < 0$, we write $\lambda = -\mu^2$ and determine the general solution of (8.26) to be

$$X(x) = A\cosh\mu x + B\sinh\mu x,$$

so that taking the derivative

$$X'(x) = \mu A\sinh\mu x + \mu B\cosh\mu x.$$

Applying the boundary condition $X'(0) = 0$ yields $B = 0$. The boundary condition $X'(L) = 0$ then yields

$$\mu A\sinh\mu L = 0,$$

which for $\mu \neq 0$ has only the solution $A = 0$. Therefore, there is no nontrivial solution for u with $\lambda < 0$. Finally, for $\lambda = 0$, the general solution of (8.26) is

$$X(x) = A + Bx,$$

so that taking the derivative

$$X'(x) = B.$$

The boundary condition $X'(0) = 0$ yields $B = 0$; $X'(L) = 0$ is then trivially satisfied. Therefore, we have an additional eigenvalue and eigenfunction given by

$$\lambda_0 = 0, \quad X_0(x) = 1,$$

which can be seen as extending the formula obtained for eigenvalues and eigenvectors for positive λ given by (8.27) and (8.28) to $n = 0$.

We now turn to the equation for $T(t)$. The equation corresponding to eigenvalue λ_n, using (8.27), is given by

$$T' + \left(n^2\pi^2 D/L^2\right) T = 0,$$

which has solution proportional to

$$T_n = e^{-n^2\pi^2 Dt/L^2}, \tag{8.29}$$

valid for $n = 0, 1, 2, \ldots$. Therefore, multiplying the solutions given by (8.28) and (8.29), we conclude that the functions

$$u_n(x,t) = \cos(n\pi x/L)e^{-n^2\pi^2 Dt/L^2} \tag{8.30}$$

satisfy the pde given by (8.12) and the boundary conditions given by (8.25) for every nonnegative integer n.

The principle of linear superposition then yields the general solution as

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} c_n u_n(x,t) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L)e^{-n^2\pi^2 Dt/L^2}, \end{aligned} \tag{8.31}$$

where we have redefined the constants so that $c_0 = a_0/2$ and $c_n = a_n$, $n = 1, 2, 3, \ldots$. The final solution step is to satisfy the initial conditions given by (8.13). At $t = 0$, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L), \tag{8.32}$$

which we recognize as a Fourier cosine series (8.9) for an even function $f(x)$ with period $2L$. We have obtained a cosine series for the periodic extension of $f(x)$ because of the boundary condition $f'(0) = 0$, which is satisfied by an even function with continuous first derivative. From our solution (8.8) for the coefficients of a Fourier cosine series, we determine

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx. \tag{8.33}$$

Thus the solution to the diffusion equation with homogenous Neumann boundary conditions defined by (8.12), (8.13) and (8.25) is given by (8.31) with the coefficients computed from (8.33).

Example: Determine the concentration of a dye in a pipe of length L, where the dye has total initial mass M_0 and is initially concentrated at the center of the pipe, and the ends of the pipe are sealed

Again we model the initial concentration of the dye by a delta-function centered at $x = L/2$. From (8.33),

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L M_0\delta(x-\frac{L}{2})\cos\frac{n\pi x}{L}dx \\ &= \frac{2M_0}{L}\cos\left(n\pi/2\right) \\ &= \begin{cases} 2M_0/L & \text{if } n = 0,4,8,\ldots; \\ -2M_0/L & \text{if } n = 2,6,10,\ldots; \\ 0 & \text{if } n = 1,3,5,\ldots. \end{cases} \end{aligned}$$

The first two terms in the series for $u(x,t)$ are given by

$$u(x,t) = \frac{M_0}{L}\left[1 - 2\cos\left(2\pi x/L\right)e^{-4\pi^2Dt/L^2} + \ldots\right].$$

Notice that as $t \to \infty$, $u(x,t) \to M_0/L$: the dye mass is conserved in the pipe (since the pipe ends are sealed) and eventually diffused uniformly throughout the pipe of length L.

8.6 Solution of the wave equation

8.6.1 Plucked string

We assume an elastic string with fixed ends is plucked like a guitar string. The governing equation for $u(x,t)$, the position of the string from its equilibrium position, is the wave equation

$$u_{tt} = c^2u_{xx}, \tag{8.34}$$

with $c^2 = T/\rho$ and with boundary conditions at the string ends located at $x = 0$ and L given by

$$u(0,t) = 0, \quad u(L,t) = 0. \tag{8.35}$$

Since the wave equation is second-order in time, initial conditions are required for both the displacement of the string due to the plucking and the initial velocity of the displacement. We assume

$$u(x,0) = f(x), \quad u_t(x,0) = 0, \quad 0 \le x \le L. \tag{8.36}$$

Again we use the method of separation of variables and try the ansatz

$$u(x,t) = X(x)T(t). \tag{8.37}$$

Substitution of our ansatz (8.37) into the wave equation (8.34) and separating variables results in

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\lambda,$$

yielding the two ordinary differential equations

$$X'' + \lambda X = 0, \quad T'' + \lambda c^2T = 0. \tag{8.38}$$

We solve first the equation for $X(x)$. The appropriate boundary conditions for X are given by

$$X(0) = 0, \quad X(L) = 0, \tag{8.39}$$

and we have solved this equation for $X(x)$ previously in §8.5.1 (see (8.17)). A nontrivial solution exists only when $\lambda > 0$, and our previously determined solution was

$$\lambda_n = (n\pi/L)^2, \quad n = 1, 2, 3, \ldots, \tag{8.40}$$

with corresponding eigenfunctions

$$X_n = \sin(n\pi x/L). \tag{8.41}$$

With λ_n specified, the T equation then becomes

$$T_n'' + \frac{n^2\pi^2c^2}{L^2}T_n = 0,$$

with general solution given by

$$T_n(t) = A\cos\frac{n\pi ct}{L} + B\sin\frac{n\pi ct}{L}. \tag{8.42}$$

The second of the initial conditions given by (8.36) implies

$$u_t(x, 0) = X(x)T'(0) = 0,$$

which can be satisfied only if $T'(0) = 0$. Applying this boundary condition to (8.42), we find $B = 0$. Combining our solution for $X_n(x)$, (8.41), and $T_n(t)$, we have determined that

$$u_n(x, t) = \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}, \quad n = 1, 2, 3, \ldots$$

satisfies the wave equation, the boundary conditions at the string ends, and the assumption of zero initial string velocity. Linear superposition of these solutions results in the general solution for $u(x, t)$ of the form

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}. \tag{8.43}$$

The remaining condition to satisfy is the initial displacement of the string, the first equation of (8.36). We have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L),$$

which is observed to be a Fourier sine series (8.11) for an odd function with period $2L$. Therefore, the coefficients b_n are given by (8.10),

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots. \tag{8.44}$$

Our solution to the wave equation with plucked string is thus given by (8.43) and (8.44). Notice that the solution is time periodic with period $2L/c$. The corresponding fundamental frequency is the reciprocal of the period and is given by $f = c/2L$. From our derivation of the wave equation in §8.2, the velocity c is related to the

density of the string ρ and tension of the string T by $c^2 = T/\rho$. Therefore, the fundamental frequency (pitch) of our "guitar string" increases (is raised) with increasing tension, decreasing string density, and decreasing string length. Indeed, these are exactly the parameters used to construct, tune and play a guitar.

The wave nature of our solution and the physical significance of the velocity c can be made more transparent if we make use of the trigonometric identity

$$\sin x \cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right).$$

With this identity, our solution (8.43) can be rewritten as

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} b_n \left(\sin\frac{n\pi(x+ct)}{L} + \sin\frac{n\pi(x-ct)}{L}\right). \tag{8.45}$$

The first and second sine functions can be interpreted as a traveling wave moving to the left or the right with velocity c. This can be seen by incrementing time, $t \to t+\delta$, and observing that the value of the first sine function is unchanged provided the position is shifted by $x \to x - c\delta$, and the second sine function is unchanged provided $x \to x + c\delta$. Two waves travelling in opposite directions with equal amplitude results in a *standing wave*.

8.6.2 Hammered string

In contrast to a guitar string that is plucked, a piano string is hammered. The appropriate initial conditions for a piano string would be

$$u(x,0) = 0, \quad u_t(x,0) = g(x), \quad 0 \le x \le L. \tag{8.46}$$

Our solution proceeds as previously, except that now the homogeneous initial condition on $T(t)$ is $T(0) = 0$, so that $A = 0$ in (8.42). The wave equation solution is therefore

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}. \tag{8.47}$$

Imposition of initial conditions then yields

$$g(x) = \frac{\pi c}{L}\sum_{n=1}^{\infty} n b_n \sin\frac{n\pi x}{L}.$$

The coefficient of the Fourier sine series for $g(x)$ is seen to be $n\pi c b_n/L$, and we have

$$\frac{n\pi c b_n}{L} = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx,$$

or

$$b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}dx.$$

8.6.3 General initial conditions

If the initial conditions on $u(x,t)$ are generalized to

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le L, \tag{8.48}$$

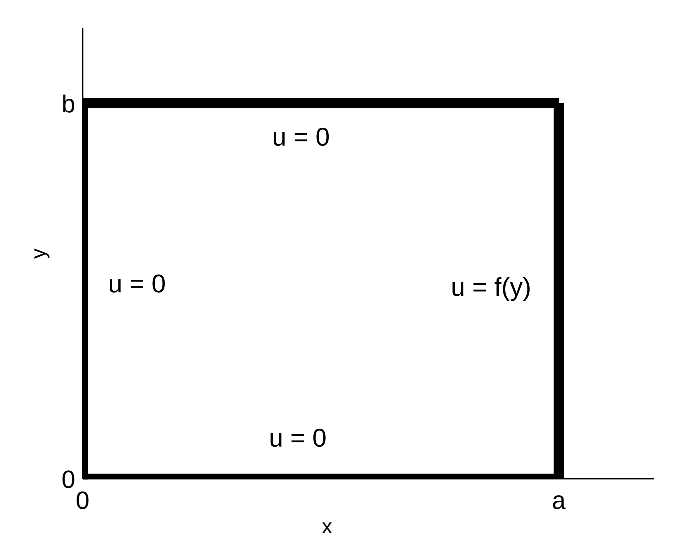


Figure 8.4: *Dirichlet problem for the Laplace equation in a rectangle.*

then the solution to the wave equation can be determined using the principle of linear superposition. Suppose $v(x,t)$ is the solution to the wave equation with initial condition (8.36) and $w(x,t)$ is the solution to the wave equation with initial conditions (8.46). Then we have

$$u(x,t) = v(x,t) + w(x,t),$$

since $u(x,t)$ satisfies the wave equation, the boundary conditions, and the initial conditions given by (8.48).

8.7 The Laplace equation

The diffusion equation in two spatial dimensions is

$$u_t = D(u_{xx} + u_{yy}).$$

The steady-state solution, approached asymptotically in time, has $u_t = 0$ so that the steady-state solution $u = u(x,y)$ satisfies the two-dimensional Laplace equation

$$u_{xx} + u_{yy} = 0. \tag{8.49}$$

We will consider the mathematical problem of solving the two dimensional Laplace equation inside a rectangular or a circular boundary. The value of $u(x,y)$ will be specified on the boundaries, defining this problem to be of Dirichlet type.

8.7.1 Dirichlet problem for a rectangle

We consider the Laplace equation (8.49) for the interior of a rectangle $0 < x < a$, $0 < y < b$, (see Fig. 8.4), with boundary conditions

$$\begin{aligned} &u(x,0) = 0, \qquad &&u(x,b) = 0, \qquad &&0 < x < a; \\ &u(0,y) = 0, \qquad &&u(a,y) = f(y), \qquad &&0 \le y \le b. \end{aligned}$$

More general boundary conditions can be solved by linear superposition of solutions.

We take our usual ansatz

$$u(x,y) = X(x)Y(y),$$

and find after substitution into (8.49),

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda,$$

with λ the separation constant. We thus obtain the two ordinary differential equations

$$X'' - \lambda X = 0, \quad Y'' + \lambda Y = 0.$$

The homogeneous boundary conditions are $X(0) = 0$, $Y(0) = 0$ and $Y(b) = 0$. We have already solved the equation for $Y(y)$ in §8.5.1, and the solution yields the eigenvalues

$$\lambda_n = \left(\frac{n\pi}{b}\right)^2, \quad n = 1, 2, 3, \ldots,$$

with corresponding eigenfunctions

$$Y_n(y) = \sin\frac{n\pi y}{b}.$$

The remaining X equation and homogeneous boundary condition is therefore

$$X'' - \frac{n^2\pi^2}{b^2}X = 0, \quad X(0) = 0,$$

and the solution is the hyperbolic sine function

$$X_n(x) = \sinh\frac{n\pi x}{b},$$

times a constant. Writing $u_n = X_n Y_n$, multiplying by a constant and summing over n, yields the general solution

$$u(x,y) = \sum_{n=0}^{\infty} c_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}.$$

The remaining inhomogeneous boundary condition $u(a,y) = f(y)$ results in

$$f(y) = \sum_{n=0}^{\infty} c_n \sinh\frac{n\pi a}{b}\sin\frac{n\pi y}{b},$$

which we recognize as a Fourier sine series for an odd function with period $2b$, and coefficient $c_n \sinh(n\pi a/b)$. The solution for the coefficient is given by

$$c_n = \frac{2}{b\sinh\frac{n\pi a}{b}}\int_0^b f(y)\sin\frac{n\pi y}{b}dy.$$

8.7.2 Dirichlet problem for a circle

The Laplace equation is commonly written symbolically as

$$\nabla^2 u = 0, \tag{8.50}$$

where ∇^2 is called the Laplacian, sometimes denoted as Δ. The Laplacian can be written in various coordinate systems, and the choice of coordinate systems usually depends on the geometry of the boundaries. Indeed, the Laplace equation is known to be separable in 13 different coordinate systems! We have solved the Laplace equation in two dimensions, with boundary conditions specified on a rectangle, using

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Here we consider boundary conditions specified on a circle, and write the Laplacian in polar coordinates by changing variables from cartesian coordinates. Polar coordinates is defined by the transformation $(r, \theta) \to (x, y)$:

$$x = r\cos\theta, \quad y = r\sin\theta; \tag{8.51}$$

and the chain rule gives for the partial derivatives

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}. \tag{8.52}$$

After taking the partial derivatives of x and y using (8.51), we can write the transformation (8.52) in matrix form as

$$\begin{pmatrix} \partial u/\partial r \\ \partial u/\partial \theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \end{pmatrix}. \tag{8.53}$$

Inversion of (8.53) can be determined from the following result, commonly proved in a linear algebra class. If

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det \mathrm{A} \neq 0,$$

then

$$\mathrm{A}^{-1} = \frac{1}{\det \mathrm{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Therefore, since the determinant of the 2×2 matrix in (8.53) is r, we have

$$\begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta/r \\ \sin\theta & \cos\theta/r \end{pmatrix} \begin{pmatrix} \partial u/\partial r \\ \partial u/\partial \theta \end{pmatrix}. \tag{8.54}$$

Rewriting (8.54) in operator form, we have

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}. \tag{8.55}$$

To find the Laplacian in polar coordinates with minimum algebra, we combine (8.55) using complex variables as

$$\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right), \tag{8.56}$$

so that the Laplacian may be found by multiplying both sides of (8.56) by its complex conjugate, taking care with the computation of the derivatives on the right-hand-side:

$$\begin{aligned}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} &= e^{i\theta}\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial \theta}\right)e^{-i\theta}\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial \theta}\right)\\ &= \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.\end{aligned}$$

We have therefore determined that the Laplacian in polar coordinates is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \tag{8.57}$$

which is sometimes written as

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

We now consider the solution of the Laplace equation in a circle with radius $r < a$ subject to the boundary condition

$$u(a,\theta) = f(\theta), \quad 0 \le \theta \le 2\pi. \tag{8.58}$$

An additional boundary condition due to the use of polar coordinates is that $u(r,\theta)$ is periodic in θ with period 2π. Furthermore, we will also assume that $u(r,\theta)$ is finite within the circle.

The method of separation of variables takes as our ansatz

$$u(r,\theta) = R(r)\Theta(\theta),$$

and substitution into the Laplace equation (8.50) using (8.57) yields

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0,$$

or

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda,$$

where λ is the separation constant. We thus obtain the two ordinary differential equations

$$r^2R'' + rR' - \lambda R = 0, \quad \Theta'' + \lambda\Theta = 0.$$

The Θ equation is solved assuming periodic boundary conditions with period 2π. If $\lambda < 0$, then no periodic solution exists. If $\lambda = 0$, then Θ can be constant. If $\lambda = \mu^2 > 0$, then

$$\Theta(\theta) = A\cos\mu\theta + B\sin\mu\theta,$$

and the requirement that Θ is periodic with period 2π forces μ to be an integer. Therefore,

$$\lambda_n = n^2, \quad n = 0,1,2,\ldots,$$

with corresponding eigenfunctions

$$\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta.$$

The R equation for each eigenvalue λ_n then becomes

$$r^2 R'' + rR' - n^2 R = 0, \tag{8.59}$$

which is an Euler equation. With the ansatz $R = r^s$, (8.59) reduces to the algebraic equation $s(s-1) + s - n^2 = 0$, or $s^2 = n^2$. Therefore, $s = \pm n$, and there are two real solutions when $n > 0$ and degenerate solutions when $n = 0$. When $n > 0$, the solution for R(r) is

$$R_n(r) = Ar^n + Br^{-n}.$$

The requirement that $u(r, \theta)$ is finite in the circle forces $B = 0$ since r^{-n} becomes unbounded as $r \to 0$. When $n = 0$, the solution for $R(r)$ is

$$R_n(r) = A + B \ln r,$$

and again finite u in the circle forces $B = 0$. Therefore, the solution for $n = 0, 1, 2, \ldots$ is $R_n = r^n$. Thus the general solution for $u(r, \theta)$ may be written as

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta), \tag{8.60}$$

where we have separated out the $n = 0$ solution to write our solution in a form similar to the standard Fourier series given by (8.2). The remaining boundary condition (8.58) specifies the values of u on the circle of radius a, and imposition of this boundary condition results in

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta). \tag{8.61}$$

Equation (8.61) is a Fourier series for the periodic function $f(\theta)$ with period 2π, i.e., $L = \pi$ in (8.2). The Fourier coefficients $a^n A_n$ and $a^n B_n$ are therefore given by (8.6) and (8.7) to be

$$\begin{aligned} a^n A_n &= \frac{1}{\pi} \int_0^{2\pi} f(\phi) \cos n\phi d\phi, \quad n = 0, 1, 2, \ldots; \\ a^n B_n &= \frac{1}{\pi} \int_0^{2\pi} f(\phi) \sin n\phi d\phi, \quad n = 1, 2, 3, \ldots, \end{aligned} \tag{8.62}$$

where we have used ϕ for the dummy variable of integration.

A remarkable fact is that the infinite series solution for $u(r, \theta)$ can be summed explicitly. Substituting (8.62) into (8.60), we obtain

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} d\phi f(\phi) \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^n (\cos n\theta \cos n\phi + \sin n\theta \sin n\phi) \right] \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\phi f(\phi) \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^n \cos n(\theta - \phi) \right]. \end{aligned}$$

We can sum the infinite series by writing $2 \cos n(\theta - \phi) = e^{in(\theta - \phi)} + e^{-in(\theta - \phi)}$, and

using the sum of the geometric series $\sum_{n=1}^{\infty} z^n = z/(1-z)$ to obtain

$$\begin{aligned}
1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi) &= 1 + \sum_{n=1}^{\infty}\left(\frac{re^{i(\theta-\phi)}}{a}\right)^n + \sum_{n=1}^{\infty}\left(\frac{re^{-i(\theta-\phi)}}{a}\right)^n \\
&= 1 + \left(\frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} + \text{c.c.}\right) \\
&= \frac{a^2-r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}.
\end{aligned}$$

Therefore,

$$u(r,\theta) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi} \frac{f(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2} d\phi,$$

an integral result for $u(r,\theta)$ known as Poisson's formula. As a trivial example, consider the solution for $u(r,\theta)$ if $f(\theta) = F$, a constant. Clearly, $u(r,\theta) = F$ satisfies both the Laplace equation and the boundary conditions so must be the solution. You can verify that $u(r,\theta) = F$ is indeed the solution by showing that

$$\int_0^{2\pi} \frac{d\phi}{a^2 - 2ar\cos(\theta-\phi) + r^2} = \frac{2\pi}{a^2 - r^2}.$$

8.8 The Schrödinger equation

The Schrödinger equation is the equation of motion for nonrelativistic quantum mechanics. This equation is a linear partial differential equation and in simple situations can be solved using the technique of separation of variables. Luckily, one of the cases that can be solved analytically is the hydrogen atom. It can be argued that the greatest success of classical mechanics was the solution of the Earth-Sun system. Similarly, it can be argued that the greatest success of quantum mechanics was the solution of the electron-proton system. The mathematical solutions of these classical and quantum two-body problems rank among the highest achievements of mankind.

8.8.1 Heuristic derivation of the Schrödinger equation

Nature consists of waves and particles. Early in the twentieth century, light was discovered to act as both a wave and a particle, and quantum mechanics assumes that matter can also act in both ways.

In general, waves are described by their wavelength λ and frequency ν, or equivalently their wavenumber $k = 2\pi/\lambda$ and their angular frequency $\omega = 2\pi\nu$. The Planck-Einstein relations for light postulated that the energy E and momentum p of a light particle, called a photon, is proportional to the angular frequency ω and wavenumber k of the light wave. The proportionality constant is called "h bar", denoted as $\hbar$, and is related to the original Planck's constant h by $\hbar = h/2\pi$. The Planck-Einstein relations are given by

$$E = \hbar\omega, \quad p = \hbar k.$$

De Broglie in 1924 postulated that matter also follows these relations.

A heuristic derivation of the Schrödinger equation for a particle of mass m and momentum p constrained to move in one dimension begins with the classical equation

$$\frac{p^2}{2m} + V(x,t) = E, \tag{8.63}$$

where $p^2/2m$ is the kinetic energy of the mass, $V(x,t)$ is the potential energy, and E is the total energy.

In search of a wave equation, we consider how to write a free wave in one dimension. Using a real function, we could write

$$\Psi = A\cos{(kx - \omega t + \phi)}, \tag{8.64}$$

where A is the amplitude and ϕ is the phase. Or using a complex function, we could write

$$\Psi = Ce^{i(kx-\omega t)}, \tag{8.65}$$

where C is a complex number containing both amplitude and phase.

We now rewrite the classical energy equation (8.63) using the Planck-Einstein relations. After multiplying by a wavefunction, we have

$$\frac{\hbar^2}{2m}k^2\Psi(x,t) + V(x,t)\Psi(x,t) = \hbar\omega\Psi(x,t).$$

We would like to replace k and ω, which refer to the wave characteristics of the particle, by differential operators acting on the wavefunction $\Psi(x,t)$. If we consider $V(x,t) = 0$ and the free particle wavefunctions given by (8.64) and (8.65), it is easy to see that to replace both k^2 and ω by derivatives we need to use the complex form of the wavefunction and explicitly introduce the imaginary unit i, that is

$$k^2 \to -\frac{\partial^2}{\partial x^2}, \quad \omega \to i\frac{\partial}{\partial t}.$$

Doing this, we obtain the intrinsically complex equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x,t)\Psi = i\hbar\frac{\partial\Psi}{\partial t},$$

which is the one-dimensional Schrödinger equation for a particle of mass m in a potential $V = V(x,t)$. This equation is easily generalized to three dimensions and takes the form

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{x},\mathbf{t}) + V(\mathbf{x},t)\Psi(\mathbf{x},t) = i\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t}, \tag{8.66}$$

where in Cartesian coordinates the Laplacian ∇^2 is written as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The Born interpretation of the wavefunction states that $|\Psi(\mathbf{x},t)|^2$ is the probability density function of the particle's location. That is, the spatial integral of $|\Psi(\mathbf{x},t)|^2$ over a volume V gives the probability of finding the particle in V at time t. Since the particle must be somewhere, the wavefunction for a bound particle is usually normalized so that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Psi(x,y,z;t)|^2 dx\,dy\,dz = 1.$$

The simple requirements that the wavefunction be normalizable as well as single valued admits an analytical solution of the Schrödinger equation for the hydrogen atom.

8.8.2 The time-independent Schrödinger equation

The space and time variables of the time-dependent Schrödinger equation (8.66) can be separated provided the potential function $V(\mathbf{x}, t) = V(\mathbf{x})$ is independent of time. We try $\Psi(\mathbf{x}, t) = \psi(\mathbf{x})f(t)$ and obtain

$$-\frac{\hbar^2}{2m} f \nabla^2 \psi + V(\mathbf{x})\psi f = i\hbar \psi f'.$$

Dividing by ψf, the equation separates as

$$-\frac{\hbar^2}{2m} \frac{\nabla^2 \psi}{\psi} + V(\mathbf{x}) = i\hbar \frac{f'}{f}.$$

The left-hand side is independent of t, the right-hand side is independent of $\mathbf{x}$, so both the left- and right-hand sides must be independent of $\mathbf{x}$ and t and equal to a constant. We call this separation constant E, thereby reintroducing the total energy into the equation. We now have the two differential equations

$$f' = -\frac{iE}{\hbar} f, \quad -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\mathbf{x})\psi = E\psi.$$

The second equation is called the time-independent Schrödinger equation. The first equation can be easily integrated to obtain

$$f(t) = e^{-iEt/\hbar},$$

which can be multiplied by a arbitrary constant.

8.8.3 Particle in a one-dimensional box

We assume that a particle of mass m is able to move freely in only one dimension and is confined to the region defined by $0 < x < L$. This is perhaps the simplest quantum mechanical problem with quantized energy levels. We take as the potential energy function

$$V(x) = \begin{cases} 0, & 0 < x < L, \\ \infty, & \text{otherwise}, \end{cases}$$

where we may assume that the particle is forbidden from the region with infinite potential energy. We will simply take as the boundary conditions on the wavefunction

$$\psi(0) = \psi(L) = 0. \tag{8.67}$$

For this potential, the time-independent Schrödinger equation for $0 < x < L$ reduces to

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi,$$

which we write in a familiar form as

$$\frac{d^2\psi}{dx^2} + \left(\frac{2mE}{\hbar^2}\right)\psi = 0. \tag{8.68}$$

Equation (8.68) together with the boundary conditions (8.67) form an ode eigenvalue problem, which is in fact identical to the problem we solved for the diffusion equation in subsection 8.5.1.

The general solution to this second-order differential equation is given by

$$\psi(x) = A\cos\frac{\sqrt{2mE}x}{\hbar} + B\sin\frac{\sqrt{2mE}x}{\hbar}.$$

The first boundary condition $\psi(0) = 0$ yields $A = 0$. The second boundary condition $\psi(L) = 0$ yields

$$\frac{\sqrt{2mE}L}{\hbar} = n\pi, \quad n = 1, 2, 3, \ldots.$$

The energy levels of the particle are therefore quantized, and the allowed values are given by

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}.$$

The corresponding wavefunction is given by $\psi_n = B\sin(n\pi x/L)$. We can normalize each wavefunction so that

$$\int_0^L |\psi_n(x)|^2 dx = 1,$$

obtaining $B = \sqrt{2/L}$. We have therefore obtained

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}, & 0 < x < L; \\ 0, & \text{otherwise.} \end{cases}$$

8.8.4 The simple harmonic oscillator

Hooke's law for a mass on a spring is given by

$$F = -Kx,$$

where K is the spring constant. The potential energy $V(x)$ in classical mechanics satisfies

$$F = -\partial V/\partial x,$$

so that the potential energy of the spring is given by

$$V(x) = \frac{1}{2}Kx^2.$$

Recall that the differential equation for a classical mass on a spring is given from Newton's law by

$$m\ddot{x} = -Kx,$$

which can be rewritten as

$$\ddot{x} + \omega^2 x = 0,$$

where $\omega^2 = K/m$. Following standard notation, we will therefore write the potential energy as

$$V(x) = \frac{1}{2}m\omega^2 x^2.$$

The time-independent Schrödinger equation for the one-dimensional simple harmonic oscillator then becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi. \tag{8.69}$$

The relevant boundary conditions are $\psi \to 0$ as $x \to \pm\infty$ so that the wavefunction is normalizable.

The Schrödinger equation given by (8.69) can be made neater by nondimensionalization. We rewrite (8.69) as

$$\frac{d^2\psi}{dx^2} + \left(\frac{2mE}{\hbar^2} - \frac{m^2\omega^2 x^2}{\hbar^2}\right)\psi = 0, \tag{8.70}$$

and observe that the dimension $[\hbar^2/m^2\omega^2] = l^4$, where l is a unit of length. We therefore let

$$x = \sqrt{\frac{\hbar}{m\omega}}y, \quad \psi(x) = u(y),$$

and (8.70) becomes

$$\frac{d^2u}{dy^2} + (\mathcal{E} - y^2)u = 0, \tag{8.71}$$

with the dimensionless energy given by

$$\mathcal{E} = 2E/\hbar\omega, \tag{8.72}$$

and boundary conditions

$$\lim_{y\to\pm\infty} u(y) = 0. \tag{8.73}$$

The dimensionless Schrödinger equation given by (8.71) together with the boundary conditions (8.73) forms another ode eigenvalue problem. A nontrivial solution for $u = u(y)$ exists only for discrete values of $\mathcal{E}$, resulting in the quantization of the energy levels. Since the second-order ode given by (8.71) has a nonconstant coefficient, we can use the techniques of Chapter 5 to find a convergent series solution for $u = u(y)$ that depends on $\mathcal{E}$. However, we will then be faced with the difficult problem of determining the values of $\mathcal{E}$ for which $u(y)$ satisfies the boundary conditions (8.73).

A path to an analytical solution can be discovered if we first consider the behavior of u for large values of y. With $y^2 >> \mathcal{E}$, (8.71) reduces to

$$\frac{d^2u}{dy^2} - y^2u = 0. \tag{8.74}$$

To determine the behavior of u for large y, we try the ansatz

$$u(y) = e^{ay^2}.$$

We have

$$u'(y) = 2aye^{ay^2},$$

$$u''(y) = e^{ay^2}\left(2a + 4a^2y^2\right) \approx 4a^2y^2e^{ay^2},$$

and substitution into (8.74) results in

$$(4a^2 - 1)y^2 = 0,$$

yielding $a = \pm 1/2$. Therefore at large y, $u(y)$ either grows like $e^{y^2/2}$ or decays like $e^{-y^2/2}$. Here, we have neglected a possible polynomial factor in front of the exponential functions. Clearly, the boundary conditions forbid the growing behavior and only allow the decaying behavior.

We proceed further by letting

$$u(y) = H(y)e^{-y^2/2},$$

and determining the differential equation for $H(y)$. After some simple calculation, we have

$$u'(y) = (H' - yH)e^{-y^2/2}, \quad u''(y) = \left(H'' - 2yH' + (y^2 - 1)H\right)e^{-y^2/2}.$$

Substitution of the second derivative and the function into (8.71) results in the differential equation

$$H'' - 2yH' + (\mathcal{E} - 1)H = 0. \tag{8.75}$$

We now solve (8.75) by a power-series ansatz. We try

$$H(y) = \sum_{k=0}^{\infty} a_k y^k.$$

Substitution into (8.75) and shifting indices as detailed in Chapter 5 results in

$$\sum_{k=0}^{\infty}\left((k+2)(k+1)a_{k+2} + (\mathcal{E} - 1 - 2k)a_k\right)y^k = 0.$$

We have thus obtained the recursion relation

$$a_{k+2} = \frac{(1+2k) - \mathcal{E}}{(k+2)(k+1)}a_k, \quad k = 0, 1, 2, \ldots. \tag{8.76}$$

Now recall that apart from a possible multiplicative polynomial factor, $u(y) \sim e^{y^2/2}$ or $e^{-y^2/2}$. Therefore, $H(y)$ either goes like a polynomial times e^{y^2} or a polynomial. The function $H(y)$ will be a polynomial only if $\mathcal{E}$ takes on specific values which truncate the infinite power series.

Before we jump to this conclusion, I want to show that if the power series does not truncate, then $H(y)$ does indeed grow like e^{y^2} for large y. We first write the Taylor series for e^{y^2}:

$$\begin{aligned} e^{y^2} &= 1 + y^2 + \frac{y^4}{2!} + \frac{y^6}{3!} + \ldots \\ &= \sum_{k=0}^{\infty} b_k y^k, \end{aligned}$$

where

$$b_k = \begin{cases} \frac{1}{(k/2)!}, & \text{n even;} \\ 0, & \text{k odd.} \end{cases}$$

For large values of y, the later terms of the power series dominate the earlier terms and the behavior of the function is determined by the large k coefficients. Provided k is even, we have

$$\begin{aligned} \frac{b_{k+2}}{b_k} &= \frac{(k/2)!}{((k+2)/2)!} \\ &= \frac{1}{(k/2)+1} \\ &\sim \frac{2}{k}. \end{aligned}$$

The ratio of the coefficients would have been the same even if we had multiplied e^{y^2} by a polynomial.

The behavior of the coefficients for large k of our solution to the Schrödinger equation is given by the recursion relation (8.76, and is

$$\begin{aligned} \frac{a_{k+2}}{a_k} &= \frac{(1+2k)-\mathcal{E}}{(k+2)(k+1)} \\ &\sim \frac{2k}{k^2} \\ &= \frac{2}{k}, \end{aligned}$$

the same behavior as the Taylor series for e^{y^2}. For large y, then, the infinite power series for $H(y)$ will grow as e^{y^2} times a polynomial. Since this does not satisfy the boundary conditions for $u(y)$ at infinity, we must force the power series to truncate, which it does if we set $\mathcal{E} = \mathcal{E}_n$, where

$$\mathcal{E}_n = 1 + 2n, \quad n = 0, 1, 2, \dots,$$

resulting in quantization of the energy. The dimensional result is

$$E_n = \hbar\omega(n + \frac{1}{2}),$$

with the ground state energy level given by $E_0 = \hbar\omega/2$. The wavefunctions associated with each E_n can be determined from the power series and are the so-called Hermite polynomials times the decaying exponential factor. The constant coefficient can be determined by requiring the wavefunctions to integrate to one.

For illustration, the first two energy eigenfunctions, corresponding to the ground state and the first excited state, are given by

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}, \quad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} x e^{-m\omega x^2/2\hbar}.$$

8.8.5 Particle in a three-dimensional box

To warm up to the analytical solution of the hydrogen atom, we solve what may be the simplest three-dimensional problem: a particle of mass m able to move freely inside a cube. Here, with three spatial dimensions, the potential is given by

$$V(x, y, z) = \begin{cases} 0, & 0 < x, y, z < L, \\ \infty, & \text{otherwise.} \end{cases}$$

We may simply impose the boundary conditions

$$\psi(0,y,z) = \psi(L,y,z) = \psi(x,0,z) = \psi(x,L,z) = \psi(x,y,0) = \psi(x,y,L) = 0.$$

The Schrödinger equation for the particle inside the cube is given by

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) = E\psi.$$

We separate this equation by writing

$$\psi(x,y,z) = X(x)Y(y)Z(z),$$

and obtain

$$-\frac{\hbar^2}{2m}\left(X''YZ + XY''Z + XYZ''\right) = EXYZ.$$

Dividing by XYZ and isolating the x-dependence first, we obtain

$$-\frac{\hbar^2}{2m}\frac{X''}{X} = \frac{\hbar^2}{2m}\left(\frac{Y''}{Y} + \frac{Z''}{Z}\right) + E.$$

The left-hand side is independent of y and z and the right-hand side is independent of x so that both sides must be a constant, which we call E_x. Next isolating the y-dependence, we obtain

$$-\frac{\hbar^2}{2m}\left(\frac{Y''}{Y}\right) = \frac{\hbar^2}{2m}\left(\frac{Z''}{Z}\right) + E - E_x.$$

The left-hand side is independent of x and z and the right-hand side is independent of x and y so that both sides must be a constant, which we call E_y. Finally, we define $E_z = E - E_x - E_y$. The resulting three differential equations are given by

$$X'' + \frac{2mE_x}{\hbar^2}X = 0, \quad Y'' + \frac{2mE_y}{\hbar^2}Y = 0, \quad Z'' + \frac{2mE_z}{\hbar^2}Z = 0.$$

These are just three independent one-dimensional box equations so that the energy eigenvalue is given by

$$E_{n_x n_y n_z} = \frac{(n_x^2 + n_y^2 + n_z^2)\pi^2\hbar^2}{2mL^2}.$$

and the associated wavefunction is given by

$$\psi_{n_x n_y n_z} = \begin{cases} \left(\frac{2}{L}\right)^{3/2} \sin\frac{n\pi x}{L} \sin\frac{n\pi y}{L} \sin\frac{n\pi z}{L}, & 0 < x,y,z < L; \\ 0, & \text{otherwise}. \end{cases}$$

8.8.6 The hydrogen atom

Hydrogen-like atoms, made up of a single electron and a nucleus, are the atomic two-body problems. As is also true for the classical two-body problem, consisting of say a planet and the Sun, the atomic two-body problem can be reduced to a one-body problem by transforming to center-of-mass coordinates and defining a

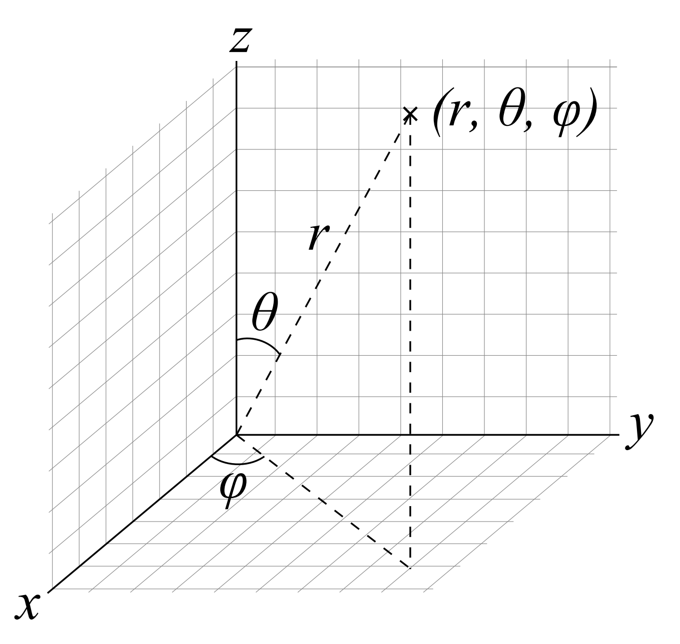


Figure 8.5: *The spherical coordinate system, with radial distance r, polar angle θ and azimuthal angle ϕ. (Public domain from Wikipedia.)*

reduced mass μ. We will not go into these details here, but will just take as the relevant Schrödinger equation

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + V(r)\psi = E\psi, \tag{8.77}$$

where the potential energy $V = V(r)$ is a function only of the distance r of the reduced mass to the center-of-mass. The explicit form of the potential energy from the electrostatic force between an electron of charge $-e$ and a nucleus of charge $+Ze$ is given by

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}. \tag{8.78}$$

With $V = V(r)$, the Schrödinger equation (8.77) is separable in spherical coordinates. With reference to Fig. 8.5, the radial distance r, polar angle θ and azimuthal angle ϕ are related to the usual cartesian coordinates by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta;$$

and by a change-of-coordinates calculation, the Laplacian can be shown to take the form

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{8.79}$$

The volume differential $d\tau$ in spherical coordinates is given by

$$d\tau = r^2\sin\theta dr d\theta d\phi. \tag{8.80}$$

A complete solution to the hydrogen atom is somewhat involved, but nevertheless is such and important and fundamental problem that I will pursue it here. Our

final result will lead us to obtain the three well-known quantum numbers of the hydrogen atom, namely the principle quantum number n, the azimuthal quantum number l, and the magnetic quantum number m.

With $\psi = \psi(r,\theta,\phi)$, we first separate out the angular dependence of the Schrödinger equation by writing

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi). \tag{8.81}$$

Substitution of (8.81) into (8.77) and using the spherical coordinate form for the Laplacian (8.79) results in

$$-\frac{\hbar^2}{2\mu}\left[\frac{Y}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] + V(r)RY = ERY.$$

To finish the separation step, we multiply by $-2\mu r^2/\hbar^2 RY$ and isolate the r-dependence on the left-hand side:

$$\frac{1}{R}\left[\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\right] + \frac{2\mu r^2}{\hbar^2}(E - V(r)) = -\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right].$$

The left-hand side is independent of θ and ϕ and the right-hand side is independent of r, so that both sides equal a constant, which we will call λ_1. The R equation is then obtained after multiplication by R/r^2:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left\{\frac{2\mu}{\hbar^2}[E - V(r)] - \frac{\lambda_1}{r^2}\right\}R = 0. \tag{8.82}$$

The Y equation is obtained after multiplication by Y:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + \lambda_1 Y = 0. \tag{8.83}$$

To further separate the Y equation, we write

$$Y(\theta,\phi) = \Theta(\theta)\Phi(\phi). \tag{8.84}$$

Substitution of (8.84) into (8.83) results in

$$\frac{\Phi}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\Theta}{\sin^2\theta}\frac{d^2\Phi}{d\phi^2} + \lambda_1\Theta\Phi = 0.$$

To finish this separation, we multiply by $\sin^2\theta/\Theta\Phi$ and isolate the θ-dependence on the left-hand side to obtain

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \lambda_1\sin^2\theta = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}.$$

The left-hand side is independent of ϕ and the right-hand side is independent of θ so that both sides equal a constant, which we will call λ_2. The Θ equation is obtained after multiplication by $\Theta/\sin^2\theta$:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\lambda_1 - \frac{\lambda_2}{\sin^2\theta}\right)\Theta = 0. \tag{8.85}$$

The Φ equation is obtained after multiplication by Φ:

$$\frac{d^2\Phi}{d\phi^2} + \lambda_2\Phi = 0. \tag{8.86}$$

The three eigenvalue ode equations for $R(r)$, $\Theta(\theta)$ and $\Phi(\phi)$ are thus given by (8.82), (8.85) and (8.86), with eigenvalues E, λ_1 and λ_2.

Boundary conditions on the wavefunction determine the allowed values for the eigenvalues. We first solve (8.86). The relevant boundary condition on $\Phi = \Phi(\phi)$ is its single valuedness, and since the azimuthal angle is a periodic variable, we have

$$\Phi(\phi + 2\pi) = \Phi(\phi). \tag{8.87}$$

Periodic solutions for $\Phi(\phi)$ are possible only if $\lambda_2 \geq 0$. We therefore obtain using the complex form the general solutions of (8.86):

$$\Phi(\phi) = \begin{cases} Ae^{i\sqrt{\lambda_2}\phi} + Be^{-i\sqrt{\lambda_2}\phi}, & \lambda_2 > 0, \\ C + D\phi, & \lambda_2 = 0. \end{cases}$$

The periodic boundary conditions given by (8.87) requires that $\sqrt{\lambda_2}$ is an integer and $D = 0$. We therefore define $\lambda_2 = m^2$, where m is any integer, and take as our eigenfunction

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi},$$

where we have nomalized Φ_m so that

$$\int_0^{2\pi} |\Phi_m(\phi)|^2 d\phi = 1.$$

The quantum number m is commonly called the magnetic quantum number because when the atom is placed in an external magnetic field, its energy levels become dependent on m.

To solve the Θ equation (8.85), we let

$$w = \cos\theta, \quad P(w) = \Theta(\theta).$$

Then

$$\sin^2\theta = 1 - w^2$$

and

$$\frac{d\Theta}{d\theta} = \frac{dP}{dw}\frac{dw}{d\theta} = -\sin\theta\frac{dP}{dw},$$

allowing us the replacement

$$\frac{d}{d\theta} = -\sin\theta\frac{d}{dw}.$$

With these substitutions and $\lambda_2 = m^2$, (8.85) becomes

$$\frac{d}{dw}\left[(1-w^2)\frac{dP}{dw}\right] + \left(\lambda_1 - \frac{m^2}{1-w^2}\right)P = 0. \tag{8.88}$$

To solve (8.88), we first consider the case $m = 0$. Expanding the derivative, (8.88) then becomes

$$(1-w^2)\frac{d^2P}{dw} - 2w\frac{dP}{dw} + \lambda_1 P = 0. \tag{8.89}$$

Since this is an ode with nonconstant coefficients, we try a power series ansatz of the usual form

$$P(w) = \sum_{k=0}^{\infty} a_k w^k. \tag{8.90}$$

Substitution of (8.90) into (8.89) yields

$$\sum_{k=2}^{\infty} k(k-1)a_k w^{k-2} - \sum_{k=0}^{\infty} k(k-1)a_k w^k - \sum_{k=0}^{\infty} 2k a_k w^k + \sum_{k=0}^{\infty} \lambda_1 a_k w^k = 0.$$

Shifting the index in the first expression and combining terms results in

$$\sum_{k=0}^{\infty} \left\{ (k+2)(k+1)a_{k+2} - \left[(k(k+1) - \lambda_1\right] a_k \right\} w^k = 0.$$

Finally, setting the coefficients of the power series equal to zero results in the recursion relation

$$a_{k+2} = \frac{k(k+1) - \lambda_1}{(k+2)(k+1)} a_k.$$

Even and odd coefficients decouple, and the even solution is of the form

$$P_{\text{even}}(w) = \sum_{n=0}^{\infty} a_{2n} w^{2n},$$

and the odd solution is of the form

$$P_{\text{odd}}(w) = \sum_{n=0}^{\infty} a_{2n+1} w^{2n+1}.$$

An application of Gauss's test for series convergence can show that both the even and the odd solutions diverge when $|w| = 1$ unless the series terminates. Since the wavefunction must be everywhere finite, the series must terminate and we obtain the discrete eigenvalues

$$\lambda_1 = l(l+1), \quad \text{for } l = 0, 1, 2, \ldots. \tag{8.91}$$

The quantum number l is commonly called the azimuthal quantum number despite having arisen from the polar angle equation. The resulting eigenfunctions $P_l(w)$ are called the Legendre polynomials. These polynomials are usually normalized such that $P_l(1) = 1$, and the first four Legendre polynomials are given by

$$P_0(w) = 1, \qquad P_1(w) = w,$$

$$P_2(w) = \frac{1}{2}(3w^2 - 1), \qquad P_3(w) = \frac{1}{2}(5w^3 - 3w).$$

With $\lambda_1 = l(l+1)$, we now reconsider (8.88). Expanding the derivative gives

$$(1 - w^2)\frac{d^2P}{dw^2} - 2w\frac{dP}{dw} + \left(l(l+1) - \frac{m^2}{1 - w^2}\right) P = 0. \tag{8.92}$$

Equation 8.92) is called the associated Legendre equation. The associated Legendre equation with $m = 0$,

$$(1 - w^2)\frac{d^2P}{dw^2} - 2w\frac{dP}{dw} + l(l+1)P = 0. \tag{8.93}$$

is called the Legendre equation. We now know that the Legendre equation has eigenfunctions given by the Legendre polynomials, $P_l(w)$. Amazingly, the eigenfunctions of the associated Legendre equation can be obtained directly from the Legendre polynomials. For ease of notation, we will assume that $m > 0$. To include the cases $m < 0$, we need only replace m everywhere by $|m|$.

To see how to obtain the eigenfunctions of the associated Legendre equation, we will first show how to derive the associated Legendre equation from the Legendre equation. We will need to differentiate the Legendre equation (8.93) m times and to do this we will make use of Leibnitz's formula for the mth derivative of a product:

$$\frac{d^m}{dx^m}\left[f(x)g(x)\right] = \sum_{j=0}^{m}\binom{m}{j}\frac{d^j f}{dx^j}\frac{d^{m-j}g}{dx^{m-j}},$$

where the binomial coefficients are given by

$$\binom{m}{j} = \frac{m!}{j!(m-j)!}.$$

We first compute

$$\frac{d^m}{dw^m}\left[(1-w^2)\frac{d^2P}{dw^2}\right] = \sum_{j=0}^{m}\binom{m}{j}\left[\frac{d^j}{dw^j}(1-w^2)\right]\left[\frac{d^{m-j}}{dw^{m-j}}\frac{d^2P}{dw^2}\right].$$

Only the terms $j = 0, 1$ and 2 contribute, and using

$$\binom{m}{0} = 1, \quad \binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2},$$

and the more compact notation

$$\frac{d^nP}{dw^n} = P^{(n)}(w),$$

we find

$$\frac{d^m}{dw^m}\left[(1-w^2)P^{(2)}\right] = (1-w^2)P^{(m+2)} - 2mwP^{(m+1)} - m(m-1)P^{(m)}. \tag{8.94}$$

We next compute

$$\frac{d^m}{dw^m}\left[2w\frac{dP}{dw}\right] = \sum_{j=0}^{m}\binom{m}{j}\left[\frac{d^j}{dw^j}2w\right]\left[\frac{d^{m-j}}{dw^{m-j}}\frac{dP}{dw}\right].$$

Here, only the terms $j = 0$ and 1 contribute, and we find

$$\frac{d^m}{dw^m}\left[2wP^{(1)}\right] = 2wP^{(m+1)} + 2mP^{(m)}. \tag{8.95}$$

Differentiating the Legendre equation (8.93) m times, using (8.94) and (8.95), and defining

$$p(w) = P^{(m)}(w)$$

results in

$$(1-w^2)\frac{d^2p}{dw^2} - 2(1+m)w\frac{dp}{dw} + \left[l(l+1) - m(m+1)\right]p = 0. \tag{8.96}$$

Finally, we define

$$q(w) = (1 - w^2)^{m/2} p(w). \tag{8.97}$$

Then using

$$\frac{dp}{dw} = (1 - w^2)^{-m/2} \left(\frac{dq}{dw} + \frac{mwq}{1 - w^2} \right) \tag{8.98}$$

and

$$\frac{d^2p}{dw^2} = (1 - w^2)^{-m/2} \left\{ \frac{d^2q}{dw^2} + \frac{2mw}{1 - w^2} \frac{dq}{dw} + \left[\frac{m(m+2)w^2}{(1 - w^2)^2} + \frac{m}{1 - w^2} \right] q \right\}, \tag{8.99}$$

we substitute (8.98) and (8.99) into (8.96) and cancel the common factor of $(1 - w^2)^{-m/2}$ to obtain

$$(1 - w^2)\frac{d^2q}{dw^2} - 2w\frac{dq}{dw} + \left[l(l+1) - \frac{m^2}{1 - w^2} \right] q = 0,$$

which is just the associated Legendre equation (8.92) for $q = q(w)$. We may call the eigenfunctions of the associated Legendre equation $P_{lm}(w)$, and with $q(w) = P_{lm}(w)$, we have determined the following relationship between the eigenfunctions of the associated Legendre equation and the Legendre polynomials:

$$P_{lm}(w) = (1 - w^2)^{|m|/2} \frac{d^{|m|}}{dw^{|m|}} P_l(w), \tag{8.100}$$

where we have now replaced m by its absolute value to include the possibility of negative integers. Since $P_l(w)$ is a polynomial of order l, the expression given by (8.100) is nonzero only when $|m| \leq l$. Sometimes the magnetic quantum number m is written as m_l to signify that its range of allowed values depends on the value of l.

At last, we need to solve the eigenvalue ode for $R = R(r)$ given by (8.82), with $V(r)$ given by (8.78) and λ_1 given by (8.91). The radial equation is now

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr} \right) + \left\{ \frac{2\mu}{\hbar^2} \left[E + \frac{Ze^2}{4\pi\epsilon_0 r} \right] - \frac{l(l+1)}{r^2} \right\} R = 0. \tag{8.101}$$

Note that each term in this equation has units of one over length squared times R and that $E < 0$ for a bound state solution.

It is customary to nondimensionalize the length scale so that $2\mu E/\hbar^2 = -1/4$ in dimensionless units. Furthermore, multiplication of $R(r)$ by r can also simplify the derivative term. To these ends, we change variables to

$$\rho = \frac{\sqrt{8\mu|E|}}{\hbar} r, \quad u(\rho) = rR(r),$$

and obtain after multiplication of the entire equation by ρ the simplified equation

$$\frac{d^2u}{d\rho^2} + \left\{ \frac{\alpha}{\rho} - \frac{1}{4} - \frac{l(l+1)}{\rho^2} \right\} u = 0, \tag{8.102}$$

where α now plays the role of a dimensionless eigenvalue, and is given by

$$\alpha = \frac{Ze^2}{4\pi\epsilon_0\hbar}\sqrt{\frac{\mu}{2|E|}}. \tag{8.103}$$

As we saw for the problem of the simple harmonic oscillator, it may be helpful to consider the behavior of $u = u(\rho)$ for large ρ. For large ρ, (8.102) simplifies to

$$\frac{d^2u}{d\rho^2} - \frac{1}{4}u = 0,$$

with two independent solutions

$$u(\rho) = \begin{cases} e^{\rho/2}, \\ e^{-\rho/2}. \end{cases}$$

Since the relevant boundary condition here must be $\lim_{\rho\to\infty} u(\rho) = 0$, only the second decaying exponential solution can be allowed.

We can also consider the behavior of (8.102) for small ρ. Multiplying by ρ^2, and neglecting terms proportional to ρ and ρ^2 that are not balanced by derivatives, results in the Cauchy-Euler equation

$$\rho^2\frac{d^2u}{d\rho^2} - l(l+1)u = 0,$$

which can be solved by the ansatz $u = \rho^s$. After canceling ρ^s, we obtain

$$s(s-1) - l(l+1) = 0,$$

which has the two solutions

$$s = \begin{cases} l+1, \\ -l. \end{cases}$$

If $R = R(r)$ is finite at $r = 0$, the relevant boundary condition here must be $\lim_{\rho\to 0} u(\rho) = 0$ so that only the first solution $u(\rho) \sim \rho^{l+1}$ can be allowed.

Combining these asymptotic results for large and small ρ, we now try to substitute

$$u(\rho) = \rho^{l+1}e^{-\rho/2}F(\rho) \tag{8.104}$$

into (8.102). After some algebra, the resulting differential equation for $F = F(\rho)$ is found to be

$$\frac{d^2F}{d\rho^2} + \left(\frac{2(l+1)}{\rho} - 1\right)\frac{dF}{d\rho} + \frac{\alpha-(l+1)}{\rho}F = 0.$$

We are now in a position to try a power-series ansatz of the form

$$F(\rho) = \sum_{k=0}^{\infty} a_k\rho^k$$

to obtain

$$\sum_{k=2}^{\infty} k(k-1)a_k\rho^{k-2} + \sum_{k=1}^{\infty} 2(l+1)ka_k\rho^{k-2} - \sum_{k=1}^{\infty} ka_k\rho^{k-1} + \sum_{k=0}^{\infty}\left[\alpha-(l+1)\right]a_k\rho^{k-1} = 0.$$

Shifting indices, bringing the lower summations down to zero by including zero terms, and finally combining terms, we obtain

$$\sum_{k=0}^{\infty}\left\{\left[(k+1)(k+2(l+1))\right]a_{k+1} - (1+l+k-\alpha)\,a_k\right\}\rho^{k-1} = 0.$$

Setting the coefficients of this power series equal to zero gives us the recursion relation

$$a_{k+1} = \frac{1+l+k-\alpha}{(k+1)\big(k+2(l+1)\big)} a_k.$$

For large k, we have $a_{k+1}/a_k \to 1/k$, which has the same behavior as the power series for e^ρ, resulting in a solution for $u = u(\rho)$ that behaves as $u(\rho) = e^{\rho/2}$ for large ρ. To exclude this solution, we must require the power series to terminate, and we obtain the discrete eigenvalues

$$\alpha = 1 + l + n', \quad n' = 0, 1, 2, \ldots .$$

The function $F = F(\rho)$ is then a polynomial of degree n' and is known as an associated Laguerre polynomial.

The energy levels of the hydrogen-like atoms are determined from the allowed α eigenvalues. Using (8.103), and defining

$$n = 1 + l + n',$$

for the nonnegative integer values of n' and l, we have

$$E_n = -\frac{\mu Z^2 e^4}{2(4\pi\epsilon_0)^2\hbar^2 n^2}, \quad n = 1, 2, 3, \ldots .$$

If we consider a specific energy level E_n, then the allowed values of the quantum number l are nonnegative and satisfy $l = n - n' - 1$. For fixed n then, the quantum number l can range from 0 (when $n' = n - 1$) to $n - 1$ (when $n' = 0$).

To summarize, there are three integer quantum numbers n, l, and m, with

$$\begin{aligned} n &= 1, 2, 3, \ldots, \\ l &= 0, 1, \ldots, n-1, \\ m &= -l, \ldots, l, \end{aligned}$$

and for each choice of quantum numbers (n, l, m) there is a corresponding energy eigenvalue E_n, which depends only on n, and a corresponding energy eigenfunction $\psi = \psi_{nlm}(r, \theta, \phi)$, which depends on all three quantum numbers.

For illustration, we exhibit the wavefunctions of the ground state and the first excited states of hydrogen-like atoms. Making use of the volume differential (8.80), normalization is such that

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} |\psi_{nlm}(r, \theta, \phi)|^2 r^2 \sin\theta \, dr \, d\theta \, d\phi = 1.$$

Using the definition of the Bohr radius as

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2},$$

the ground state wavefunction is given by

$$\psi_{100} = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0},$$

and the three degenerate first excited states are given by

$$\psi_{200} = \frac{1}{4\sqrt{2\pi}} \left(\frac{Z}{a_0}\right)^{3/2} \left(2 - \frac{Zr}{a_0}\right) e^{-Zr/2a_0},$$

$$\psi_{210} = \frac{1}{4\sqrt{2\pi}} \left(\frac{Z}{a_0}\right)^{3/2} \frac{Zr}{a_0} e^{-Zr/2a_0} \cos\theta,$$

$$\psi_{21\pm1} = \frac{1}{8\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{3/2} \frac{Zr}{a_0} e^{-Zr/2a_0} \sin\theta e^{\pm i\phi}.$$